A Flat, non-Newtonian Liquid Sheet Undergoing Linear Breakup

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Interest in the atomization of non-Newtonian liquids is growing because of applications such as medical sprays, paints and coatings, spray drying, and consumer products. However, basic treatments such as linear stability analyses of non-Newtonian flows are lacking. Our study adds to this area by developing a model for flat sheet breakup that can be used to describe drop formation in paint, medical, and consumer product sprays, along with spray drying processes. The boundary layer integral approach is used to obtain the averaged momentum equations for the sheet, with these equations then non-dimensionalized using the initial sheet thickness (h_0) and initial liquid velocity (u_0) as scaling parameters. Finally, a linear stability analysis is performed to obtain a dispersion equation. The influences of mean liquid velocity, surrounding gas velocity, and relative velocity on disturbance growth rate and wavenumber were considered. Results showed that an increase in the relative velocity between the surrounding gas and the liquid makes the sheet more unstable, i.e. leads to larger growth rates, increased wavenumber at the maximum growth rate, and a wider range of disturbance wavenumbers that can lead to sheet breakup. An increase in the mean liquid velocity also increases the maximum growth rate, increases the wavenumber at maximum growth rate, and again enlarges the range of disturbance wavenumbers that can lead to breakup. Finally, we also found that reducing the power law index makes the flow more unstable.

1. Introduction

Atomization is necessary in many industrial processes since the liquid must be dispersed throughout the operating domain. Since the first step in the atomization process is usually the breakup of a liquid sheet or jet, the wavelength for which instability-driven sheet or jet breakup occurs is a quantity of interest to practicing engineers because it can be used to design atomizers [1-3]. The breakup wavelength is usually obtained through mathematical analysis of the fluid flow. Numerous examples exist for Newtonian fluids with [4] being one example.

The first flat sheet stability investigation was performed by Squire [5] on an inviscid liquid film moving in a still environment. He found that surface tension forces always damp out perturbations. Dombrowski and Johns [6] extended Squire's [5] work by investigating the aerodynamic instability of a viscous Newtonian liquid sheet. They found that viscous forces oppose the growth of instabilities. More recently, there have been non-linear analyses describing the breakup of Newtonian liquid sheets. Rangel and Sirignano [7] developed one such model using the vortex discretization technique to study the growth of Kelvin-Helmholtz instabilities.

While the literature is full of studies that treat Newtonian liquid sheet breakup there is much less information available that describes the stability and breakup of non-Newtonian liquid sheets, probably because these analyses are physically more complex and mathematically more complicated. The studies that have been reported include: Ng and Mei [8], who studied roll waves of mud modeled as a power-law fluid and observed that the existence of long roll waves depends on the power-law index, but did not account for surface tension; Hwang *et al.* [9], who conducted a linear stability analysis on a thin film of power-law fluid flowing down an inclined plane and found that an increase in the Reynolds number destabilized the flow; and Dandapat and Mukhopadhyaya [10] who considered waves occurring at the surface of a power-law fluid film flowing down an inclined plane and observed that the power law index plays a vital role in the breakup process. All three analyses used the boundary layer integral method to derive nonlinear equations that describe the liquid evolution and then performed a linear stability analysis to arrive at a dispersion equation. However, in none of these cases did the authors consider the effects of surrounding gas motion. This factor is known to very important when accurately describing spray formation.

In summary, there is considerable literature describing Newtonian flat sheet and cylindrical jet breakup. There is a much smaller body of literature on the evolution of non-Newtonian flat sheets, all of which points out the importance of the flow behavior index in the breakup process. However, an analysis of the stability a non-Newtonian power-law liquid flat sheet in a moving gas is lacking. This study helps fill that gap.

2. Model Description

Like previous authors [8-10], we have assumed our liquid obeys the Ostwald-de-Wilde (power law) rheological model. There are two reasons for this. First, the model is mathematically straightforward. Second, the rheological behavior of a wide variety of common fluids can be described by this expression.

Model development starts by applying the long wave assumption to the governing equations, yielding the following expressions

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{1}$$

$$\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}\right]=-\frac{1}{r}\frac{\partial p}{\partial x}+\frac{1}{r}\frac{\partial t_{xy}}{\partial y} \tag{2}$$

$$\frac{1}{r}\frac{\partial p}{\partial y}=0 \tag{3}$$

The pressure gradient in (2) is eliminated by using (3) and the dynamic boundary condition. Integrating (1) and (2) with respect to y from 0 to h (i.e., through the sheet thickness) and using the fully developed velocity profile for a power-law fluid, as given by

$$u = u_0 \left[\frac{1+2n}{(u_a^* - 1)(1+2n) + n} \right] \left[u_a^* - \left\{ 1 - \left(\frac{y}{h} \right)^{\frac{1+n}{n}} \right\} \right] \tag{4}$$

yields the following averaged momentum equations

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{5}$$

$$\frac{\partial q}{\partial t} + 2b \frac{q}{h} \frac{\partial q}{\partial x} - b \left(\frac{q}{h} \right)^2 \frac{\partial h}{\partial x} = \frac{s}{h} h \frac{\partial^3 h}{\partial x^3} + \frac{m_n}{r} \left(\frac{q}{h^2} \right)^n \tag{6}$$

where u_0 is the initial mean liquid velocity, u_a* is a dimensionless surrounding gas velocity, n is the liquid flow behavior index, μ_n is the liquid consistency index, ρ and σ are the liquid density and surface tension, respectively, and

$$b = \frac{(2n+1) \left[(2n+1)(u_a^* - 1) u_a^* - u_a^* + \frac{2(n+1)^2}{(3n+2)} \right]}{\left[(u_a^* - 1)(1+2n) + n \right]^2} \tag{7}$$

and

$$q = \int_0^{h_0} u \, dy \, . \tag{8}$$

u_0 is defined as

$$u_0 = \frac{1}{h_0} \int_0^{h_0} u \, dy = \left(\frac{1}{m_n} \frac{\partial p}{\partial x} \right)^{\frac{1}{n}} \frac{n}{n+1} h^{1+n} \frac{\left[(u_a^* - 1)(1+2n) + n \right]}{1+2n} \tag{9}$$

and u_a^* as

$$u_a^* = \frac{u_a}{u_0} \frac{(1-n)}{(1+2n)} \frac{1}{[1 - \frac{u_a}{u_0} \left(\frac{n}{n+1} \right)]} \, . \tag{10}$$

The flow field is now non-dimensionalized and decomposed into steady and perturbed components, with the resulting equations then linearized

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{11}$$

$$\frac{\partial q}{\partial t} + 2b \frac{\partial q}{\partial x} - b \frac{\partial h}{\partial x} = -\frac{1}{We} \frac{\partial^3 h}{\partial x^3} + \frac{n A^n}{\mathrm{Re}} (1 + q - 2h) \tag{12}$$

Here the appropriate Weber and Reynolds are defined and A is

$$We = \frac{r h_0 u_0^2}{s} \qquad \mathrm{Re} = \frac{r h_0^n u_0^{2-n}}{m_n} \qquad A = \frac{(1+2n)(1+n)}{n[(u_a^* - 1)(1+2n) + n]}$$

Equations (11) and (12) are combined and the flow rate eliminated. This yields a single second order differential equation for h

$$\frac{\partial^2 h}{\partial t^2} + 2b \frac{\partial^2 h}{\partial x \partial t} + b \frac{\partial^2 h}{\partial x^2} + \frac{1}{We} \frac{\partial^4 h}{\partial x^4} - \frac{n A^n}{\mathrm{Re}} (2 \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}) = 0 \tag{13}$$

A traveling wave perturbation is assumed to exist and substituted into (13). This results in the following dispersion equation

$$w^2 - \left(\frac{A^n n}{\mathrm{Re}} + 2ibk \right) w + \frac{k^4}{We} - bk^2 + \frac{2i A^n n k}{\mathrm{Re}} = 0 \tag{14}$$

Unstable behaviour occurs for

$$w = \frac{-1}{2\,\mathrm{Re}}\left[\mathrm{A}^n n - \left\{\frac{1}{2}\left(a + (a^2 + b^2)^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}}\right] \tag{15}$$

where

$$a = -4k^2 b^2\,\mathrm{Re}^2 + (\mathrm{A}^n n)^2 + 4bk^2\,\mathrm{Re}^2 - \frac{4k^4\,\mathrm{Re}^2}{We} \qquad b = 4k\,\mathrm{Re}\,\mathrm{A}^n n(b-2)$$

The model developed above was validated against previously reported predictions. Substituting boundary conditions used by Squire [5], namely an inviscid Newtonian sheet moving through a still environment, we obtained the same value of maximum growth rate (k = 0.22) that he reported.

3. Results and Discussion

Fig. 1 illustrates how a change in ambient velocity (u_a) leads to changes in the growth rate when the mean liquid velocity (u_0) is fixed; decreasing u_a decreases the growth rate. The reason is that the relative velocity (u_{rel}) term in the convective part of the dispersion equation aids breakup, and an increase in u_a causes a reduction in this term. This provides not only a decrease in growth rate, but also a corresponding decrease in the wavenumber at maximum growth rate (yielding larger drops) and a reduction in the range of disturbance wavenumbers that could lead to breakup.

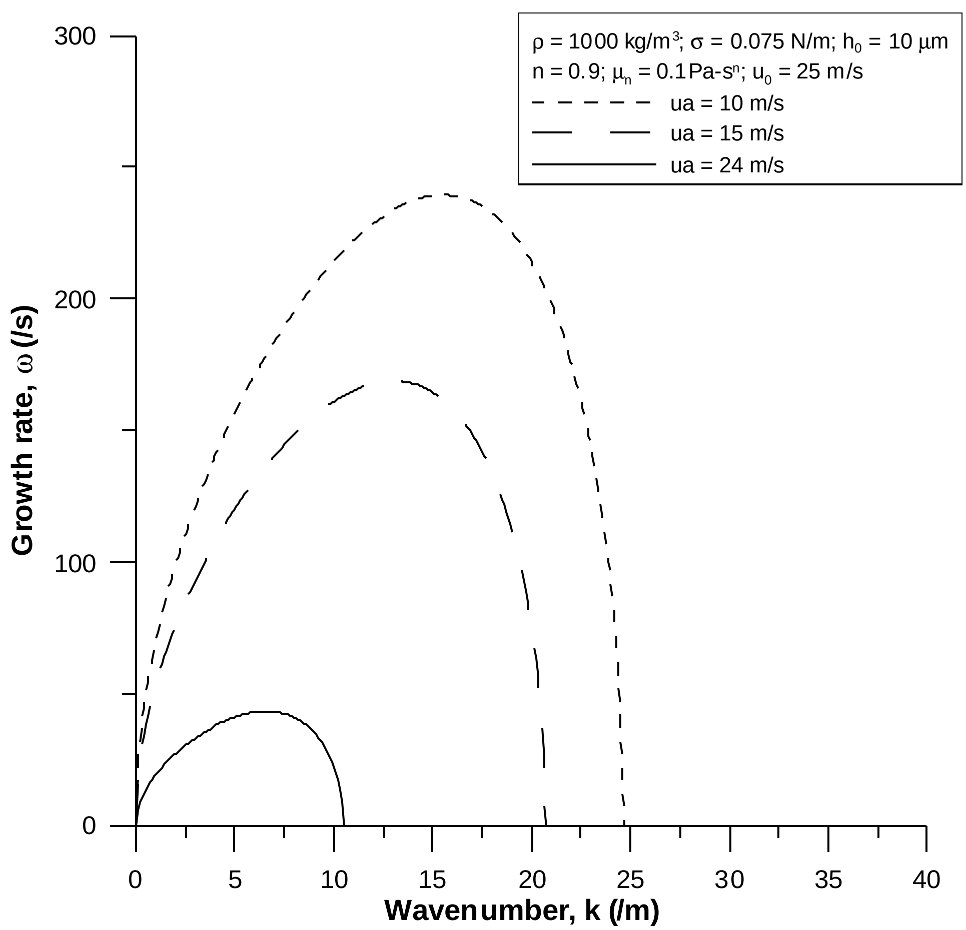

Figure 1. Growth rate (ω) versus wavenumber (k) for varying ambient velocity (u_a).

Fig. 2 demonstrates how mean liquid velocity (u_0) variations lead to changes in growth rate-wavenumber behavior; increasing u_0 increases growth rate, increases the maximum growth rate, and leads to a corresponding increase in the wavenumber at that point, plus it increases the range of wavenumbers that can lead to breakup. This is because an increase in u_0 leads to a decrease in the viscous force term of the dispersion equation (which prevents breakup) and a simultaneous increase in the convective term (which promotes breakup). Consequently, an increase in u_0 leads to smaller drop sizes.

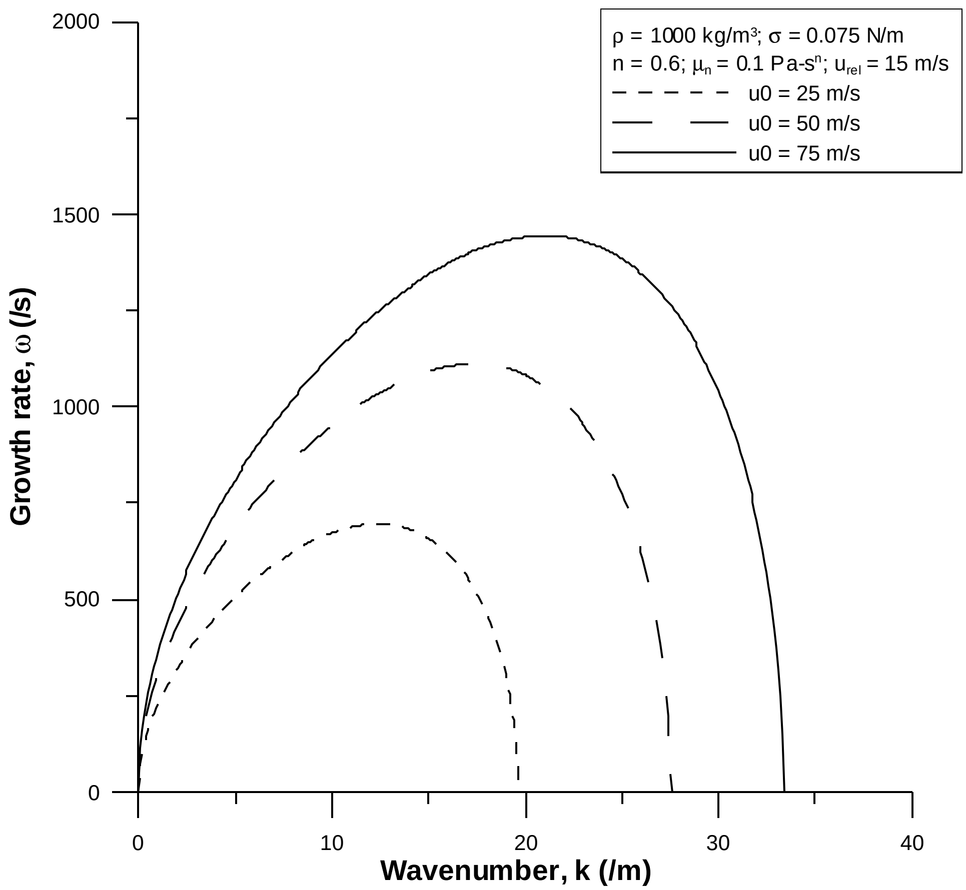

Figure 2. Growth rate (ω) versus wavenumber (k) for varying mean liquid velocity (u_0).

Comparison of Fig. 1 and Fig. 2 data for the case where u_0 = 25 m/s and u_a = 10 m/s shows us that an increase in the power law index causes a decrease in the growth rate and a decrease in the wavenumber at maximum growth rate, thus resulting in larger drops. By way of explanation, an increase in n leads to an increase in the viscous term in Eqn. (13) and a decrease in the convective term in this equation. Thus, it can be concluded that decreasing the power law index makes the flow more unstable. This is in agreement with the results of Hwang *et al*. [9].

From Fig. 3, it is evident that an increase in the relative velocity causes an increase in the maximum growth rate and an increase in the wavenumber corresponding to the maximum growth rate, as well as an increase in the range of wavelengths causing breakup. Note that the term containing the relative velocity appears in both the convective and the viscous part of Eqn. (13), but an increase in the relative velocity causes a greater increase in the convective force compared to the viscous forces. This results in an increased maximum growth rate. Hence, an increase in the relative velocity leads to smaller drops.

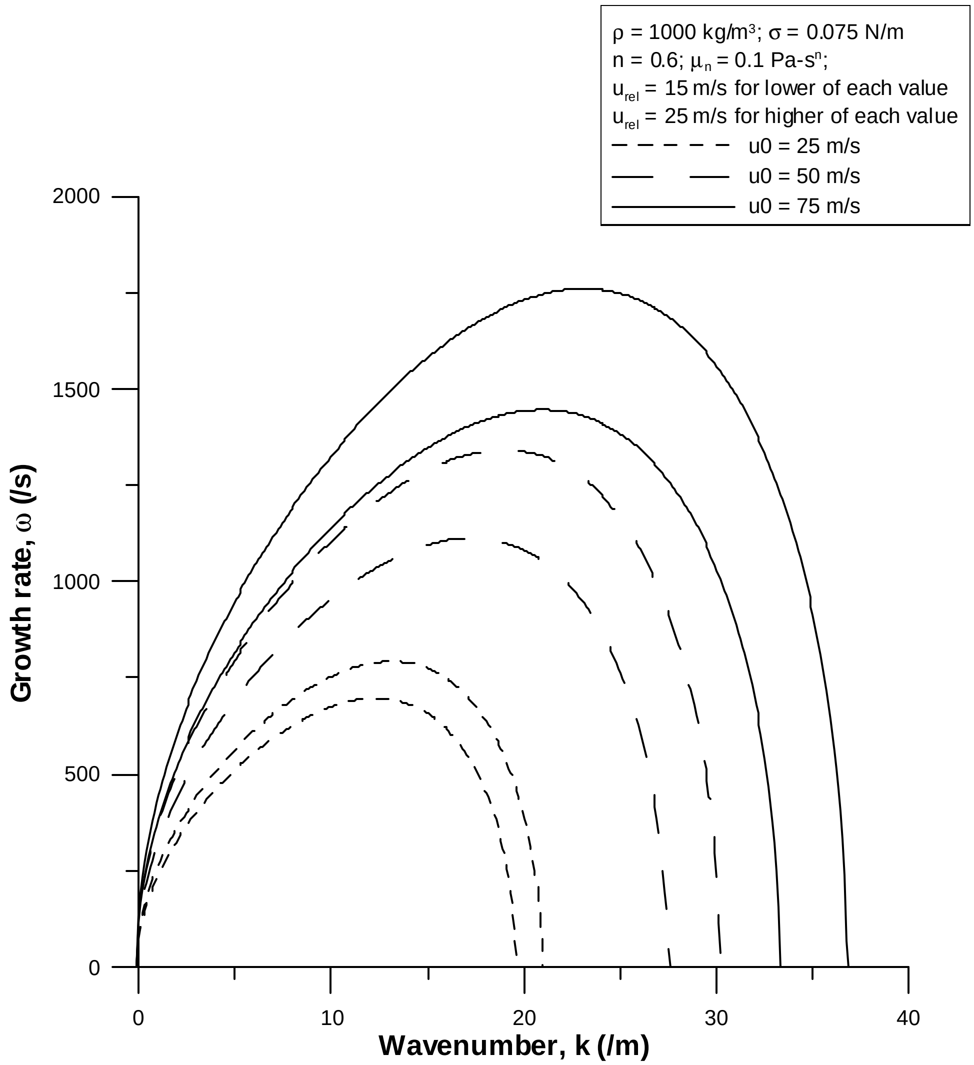

Figure 3. Growth rate (ω) versus wavenumber (k) for a constant $u_{relative}$ of 25 m/s.

A comparison of Figs. 4 and 5 show that as the consistency index decreases, the growth rate increases, the maximum growth rate increases, and the wavenumber also increases at maximum growth rate also increases. This leads to larger drop sizes. It is also shown that an decrease in the consistency index causes an increase in the range of wavelengths that could lead to breakup. This is because the consistency index appears in the viscous term and increasing n leads to an increase in the viscous term, which opposes breakup.

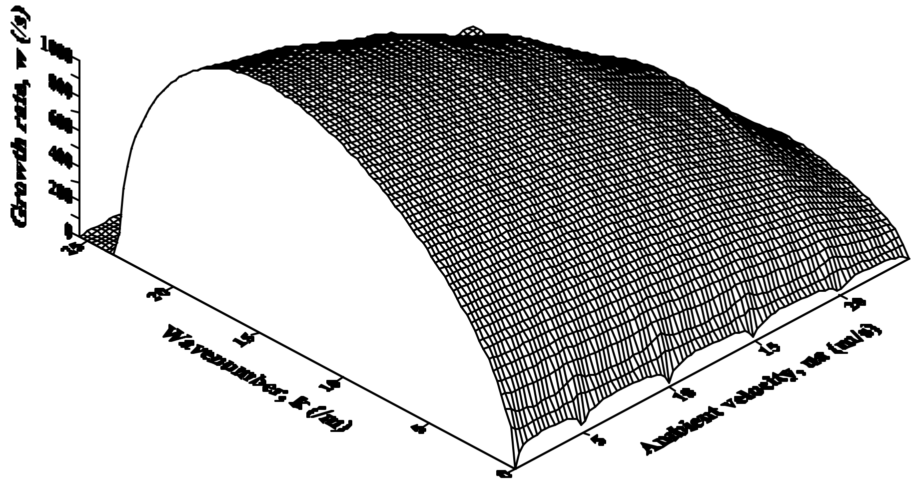

Figure 4. Growth rate (ω) versus wavenumber (k) and ambient velocity (u_a) for n = 0.75, h_0 = 10 μm, μ_n = 0.01 Pa-s^n, ρ = 1000 kg/m^3, σ = 0.075 N/m and u_0 = 25 m/s.

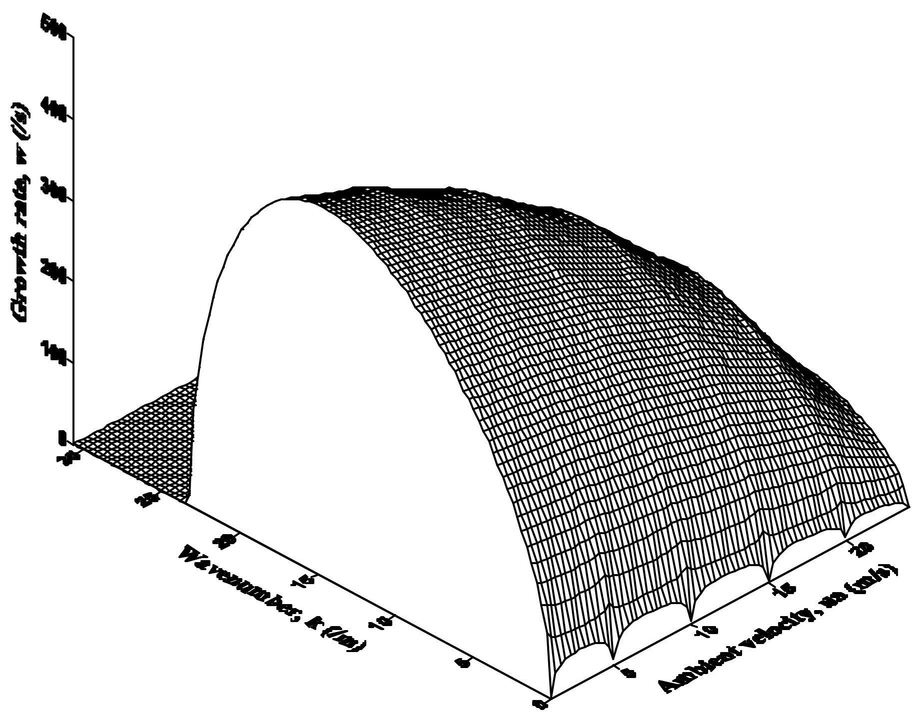

Figure 5. Growth rate (ω) versus wavenumber (k) and ambient velocity (u_a) for n = 0.75, h_0 = 10 μm, μ_n = 0.1 Pa-s^n, ρ = 1000 kg/m^3, σ = 0.075 N/m and u_0 = 25 m/s.

4. Summary and Conclusions

A linear stability analysis was performed on a flat sheet of a non-Newtonian power law fluid. A steady state, symmetric velocity profile was assumed and the effects of mean liquid (u_0), ambient gas (u_a), and relative (u_{rel}) velocities on the breakup process were examined. We note that the growth rate increases as the relative velocity increases and that increasing the relative velocity also increases the range of wavenumbers that could cause breakup. The wavenumber corresponding to the maximum growth rate increases as u_a increases for fixed u_0. An increase in u_0 at fixed u_a causes the growth rate, maximum growth rate and the corresponding wavenumber to increase, thus leading to smaller drops. An increase in u_a at fixed u_0 causes a decrease in the relative velocity, thus leading to larger drops. It is seen that decreasing the power law index makes the flow more unstable. A decrease in the consistency index has a similar effect on the flow. Note that our model is not valid at ambient gas velocities greater than the mean liquid velocity since it exhibits unphysical behavior in those cases. Future work will involve performing a non-linear analysis on the flat sheet.

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