A problem of global instability of plane liquid jets

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The mathematical definition of the modelling concerned with the global instability of a plane liquid sheet flowing in the gravitational field is developed. A particular attention is paid to the effect of the crosswise variation of the disturbances, neglected in previous studies of literature. Preliminary results show that the crosswise effects increase the properties of stability of the global modes. A rather strict comparison between numerical and analytical data obtained with exact solutions in simplified cases proves the validity of the results.

1 Introduction

The problem of the stability of two-dimensional liquid sheet flows (plane jets) subjected to surface tension in the gravitational field is of current interest in both theoretical and applied research [1]. The basic ideas explaining the physical mechanisms leading to the jet rupture (break up) seem well established [2],[3],[4], even though the attempt to recast such a level of knowledge within the framework of modern approaches of fluid dynamics instability is a current subject of research. As an example, a known experimental evidence is that the plane jet breaks up when the *flow* Weber number is lowered below unity, though, at present time, there is no theory to predict the actual value of Weber number of rupture.

Recently de Luca and Costa [5] found that for *local* Weber numbers below unity the flow experiences a *local* absolute instability, while for *local* Weber greater than unity the instability is locally convective. Of course, within the limitations of the local approach, it is not possible to predict the streamwise (vertical) location where transition from absolute to convective instability occurs, that is to say, in other words, to evaluate the *flow* Weber of rupture. On the grounds of a theory developed within the framework of the linear analysis [6], de Luca and Costa [5] hypothesised that the sheet breaks up only when the region of local absolute instability reaches a certain critical extension. This circumstance corresponds to the onset the so-called *global* modes toned at the same temporal frequency at all streamwise locations. Indeed, some experimental findings of de Luca [7] would confirm such a basic result.

However, in spite of the close agreement between theory and experiments, there are various motivations to believe that the conclusion mentioned above is still open to discussion. Basically, the non linear approach, developed more recently for a classic non parallel Ginzburg-Landau flow model [8] does not confirm the linear result; in fact, in non linear regimes (steep) global modes are triggered as soon as local linear absolute instability appears. Secondly, as far as present authors know, there is no paper in the literature dealing theoretically with the global instability study of plane liquid sheet flows falling freely in the gravitational field and subjected to the surface tension.

The above discussion constitutes the main motivation of the present work. Furthermore, one has to consider that another modern research subject is the study of the disturbances energy growth during the transient period and its connections with the combination of global modes [9]. It is known that the presence of transient growths of disturbances is to be related to the non-normal character of the operator governing the problem. Within this last context the previous analyses were conducted along the streamwise direction and a few papers only approached the effect of the crosswise one. Thus, another motivation of the present work is to investigate the *crosswise* effects on the nature of the *global* modes as well as on the possible presence of transient growth of the disturbances energy.

The present paper deals with the mathematical definition of the modelling concerned with the investigation of the global instability of a plane liquid sheet flowing in the gravitational field. A particular attention is paid towards the effect of the crosswise variation of the disturbances. In the following section it will be shown that the search for two-dimensional eigenvalues and (global) eigenfunctions can be reduced to a one-dimensional problem along the streamwise direction only, the crosswise one being accounted by the presence of a proper variable coefficient given by the crosswise eigenvalues.

The mathematical modelling is described, together with a discussion on the nature of the relevant eigenvalue problem. The paper addresses also the numerical treatment as well as a comparison with the characteristics of the deeply celebrated Ginzburg-Landau equation. Preliminary results are presented in the light of such a comparison.

2 Mathematical modelling

The model refers to a plane liquid sheet falling under gravity (2D gravitational jet) and issuing from a thin slit into a gas atmosphere. Linear theory of stability is classically developed, the basic state of flow being characterised by a velocity of the liquid dominated by inertia and gravity forces and obeying the inviscid Torricellian solution of a plane jet. No surface tension effects are considered for such a base flow. The assumptions are made that both the motions of liquid and external gas are incompressible and that the latter is also inviscid. Navier-Stokes equations are written for the perturbed velocity field together with boundary conditions on the interface, the symmetry plane for the internal liquid, and the far field for the external gas.

The general lay-out of the mathematical model are the same as described in a previous paper [10], which the interested reader is referred to for a detailed presentation. One has to stress explicitly that the analysis carried out in ref. [10] is a local-type one, leading to a multiple scales formulation in which the base flow varies along the streamwise z direction slowly with respect to the typical wavelength of disturbances; as a consequence, the amplitude of disturbances is a function of the slow scale Z, whereas a Fourier component of the wave is considered along the fast one z. The small parameter $\delta = 2gb/\overline{U}^2$, where g is the gravity acceleration, b the slit half-width and $\overline{U}$ the mean vertical velocity at the slit exit section, relates to each other the slow and the fast length scales.

In the present paper addressing the global stability investigation, on the contrary, such a scale separation is released because local wave trains at different Z locations are pieced together just by means of the relation $Z = \delta z$. In addition, as customary in global analysis, the spatial dependence on z is retained in the perturbed Navier-Stokes equations.

A convenient reformulation of the problem may be done in terms of the y lateral component of disturbances velocity v and vorticity

$$\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

where u and w are the velocity components along the longitudinal direction x and the vertical one z, respectively.

A key remark. As also noted in the previous paper [10], the initial-boundary value problem for vorticity is not coupled to the lateral velocity one, boundary conditions included. Thus, in order to illustrate the reduction technique and to give some insights about it, this paper addresses the solution of the vorticity equation only, that, from this point point of view, has to be considered as a model problem used to test the method itself.

The time evolution equation for the y vorticity is:

$$\frac{\partial \omega}{\partial t} + \overline{w}\frac{\partial \omega}{\partial z} = \frac{1}{Re}\frac{\partial^2 \omega}{\partial z^2} + \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial y^2} - \beta^2 \omega\right) \tag{1}$$

where $\overline{w} = \sqrt{1+\delta z}$ is the Torricellian solution of the base flow velocity in the absence of external gas, $Re = \overline{U}b/\nu$ is the Reynolds number of liquid motion and β is the longitudinal wavenumber of disturbances the latter having the general form:

$$\omega'(t,x,y,z) = \omega(t,y,z)e^{i\beta x} \tag{2}$$

The two boundary conditions needed to integrate the evolution equation for vorticity are

$$\omega = 0 \quad , \quad \text{for } y = 0 \tag{3}$$

$$\frac{\partial \omega}{\partial y} = 0 \quad , \quad \text{for } y = \overline{\eta}(z) \tag{4}$$

arising from symmetry considerations (for sinuous disturbances) and shear stresses vanishing at the interface $y = \overline{\eta}(z)$, where $\overline{\eta} = 1/\sqrt{1+\delta z}$

Furthermore, the present problem dealing with a global stability analysis, additional conditions are to be imposed at $z = 0$ and for sufficiently large z. In fact, the presence of the nozzle may be accounted by means of the condition that disturbances vanish at $z = 0$, and global modes are searched that vanish far downstream.

In summary, eq. (1) to be integrated over the domain $0 \le y \le \overline{\eta}$ and $0 \le z \le \infty$ represents the time evolution equation for the vorticity. Coupled to the relevant boundary conditions written above, it represents an initial value problem describing a global linear stability problem [6],[8], that in principle may be solved by the so-called direct numerical simulation technique. Since in the present problem the investigation is restricted to a linear framework, an alternative is to carry out an eigenvalues analysis of the differential operator above introduced. Indeed, this second approach will be employed in the following.

Before describing the numerical technique, it is important to note that the present problem of stability is of two-dimensional character in the sense that it includes variations of the perturbations along both streamwise and crosswise spatial co-ordinates. Problems of global stability in the literature are classically defined along the streamwise direction only; in the following section a method will be illustrated to reduce such a 2D problem to a 1D one, by means of which the crosswise effects are taken into account through a variable coefficient of the resulting ordinary differential equation. It will be shown that, indeed, such a method of variables separation yields a variable coefficient related to the eigenvalues of the differential operator along the crosswise direction.

3 Analytical and numerical techniques

It is convenient to rewrite eq. (1) as

$$\frac{\partial \omega}{\partial t} = \mathcal{L}_z(\omega) + \mathcal{L}_y(\omega) \tag{5}$$

where

$$\mathcal{L}_z \equiv -\overline{w}\frac{\partial}{\partial z} + \frac{1}{Re}\frac{\partial^2}{\partial z^2}\ ,\ \mathcal{L}_y \equiv \frac{1}{Re}\left(\frac{\partial^2}{\partial y^2} - \beta^2\right) \tag{6}$$

It is not difficult to prove that finding the eigenvalues of the $2D$ differential operator $\mathcal{L}_z + \mathcal{L}_y$ is equivalent to find the eigenvalues of the $1D$ operator

$$\frac{\partial \omega}{\partial t} = \left[\mathcal{L}_z + \gamma(z)\right](\omega) \tag{7}$$

where $\gamma(z)$ are the eigenvalues of the operator $\mathcal{L}_y$ defined on the domain $0 \leq y \leq \overline{\eta}(z)$. The latter are given by the following formula:

$$\gamma(z) = -\frac{1}{Re}\left(\beta^2 + \pi^2(1+\delta z)(n+\frac{1}{2})^2\right)\ ,\ n \in \mathbb{Z} \tag{8}$$

To summarise, the eigenvalues of the $2D$ vorticity evolution equation (1) may be evaluated as the eigenvalues of the equation:

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re}\frac{\partial^2 \omega}{\partial z^2} - \sqrt{1+\delta z}\frac{\partial \omega}{\partial z} - \frac{1}{Re}\left(\beta^2 + \pi^2(1+\delta z)(n+\frac{1}{2})^2\right)\omega \tag{9}$$

where $n \in \mathbb{Z}$. Eq. (9) is of the Ginzburg-Landau type [6] but the convective term coefficient is a variable function of z. Moreover, the coefficient of the latter term on the right hand side which is linked to the eigenvalues $\gamma(z)$, variable as well, accounts for the *crosswise* effects. If one wishes to neglect such crosswise effects, the following equation is to be solved

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re}\frac{\partial^2 \omega}{\partial z^2} - \sqrt{1+\delta z}\frac{\partial \omega}{\partial z} - \frac{1}{Re}\beta^2\omega \tag{10}$$

The aim of the present paper is to assess a basic procedure to investigate both base equations (9) and (10), from both analytical and numerical points of view. To do this, it has been convenient to modify slightly the physical functional dependence of the first derivative coefficient, which has been taken as $-(1+\delta z)$. This simplification allowed us to obtain compact analytical solutions of the equations mentioned above.

In all the cases eigenvalues and eigenfunctions, that is to say, global modes, are determined by enforcing the boundary conditions that at nozzle exit section $z = 0$ be $\omega = 0$ and that global disturbances vanish far downstream, $\omega = 0$ for $z \to \infty$.

In the case of the complete (i.e., including the cross-wise effects) equation the eigenfunctions are given by [11]

$$\omega(z) = Ke^{-z}\left(M(a,\frac{1}{2},\zeta) - \frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(1+a-b)}{\Gamma(2-b)}\ \zeta^{1-b}M(1+a-b,2-b,\zeta)\right) \tag{11}$$

where M and Γ denote Kummer M and Gamma functions [12], respectively, $\zeta = (3+\delta z)^2/(2\delta)$, a and b are two parameters, the former being related to the eigenvalue λ by the relation $a = (\lambda - 1)/(2\delta)$, the latter being equal to $1/2$. K is an undefined constant.

For the other equation, which does not take into account cross-wise effects, the eigenfunctions are given by a relationship similar to that of eq. (11), but not including the damping factor e^{-z}; furthermore, the parameters a and ζ are $a = (\lambda+1)/(2\delta)$ and $\zeta = (1+\delta z)^2/(2\delta)$, respectively.

As far as the eigenvalues λ are concerned, they may be obtained as the zero of the equation

$$\sqrt{\frac{A}{2\delta}}\Gamma\left(\frac{1}{2}\right)\Gamma\left(a+\frac{1}{2}\right)M\left(a+\frac{1}{2},\frac{3}{2},\frac{A}{2\delta}\right)-\Gamma\left(\frac{3}{2}\right)\Gamma\left(a\right)M\left(a,\frac{1}{2},\frac{A}{2\delta}\right)=0 \qquad (12)$$

where, as usual a is related to the eigenvalues themselves and A is a parameter equal to 9 for the complete equation, and 1 for the other one.

The numerical treatment of the problem has followed the implementation of a spectral algorithm based upon collocation. We used a rather standard technique, for the details of which the reader can refer to [10]. In the computations of the present work the number of collocation points is generally taken equal to 200 whereas in order to enforce the boundary condition at infinity, the truncating method has been basically employed and the length of the integration domain has been chosen equal to 500.

4 Results

Fig. 1 shows the comparison between the numerical and analytical spectra obtained for the complete simplified equation, i.e. with $1+\delta z$ in place of $\sqrt{1+\delta z}$ and $-(1+\delta z)$ as the coefficient of ω term in eq. (9). To compute these eigenvalues the following parameters have

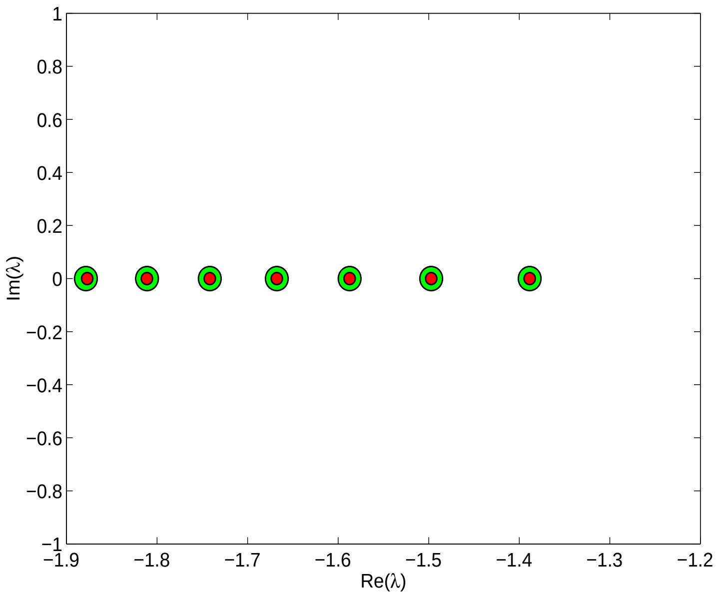

Figure 1: Spectrum in the presence of crosswise effects

been considered: $Re = 1$, $\delta = 10^{-2}$. The figure exhibits a very close agreement between the two sets of data (inner circles refer to analytical eigenvalues, outer to numerical ones); of course the accuracy of the results decreases with increasing the number of the global mode considered. From a more physical point of view, the spectrum is all confined in the left half plane, this circumstance denoting stability.

A comparison of computed and analytically evaluated first global eigenfunctions is reported in fig. 2 , that refers to the same parameters as previous fig. 1. Also in this case, one may observe a satisfactory agreement of results (dots are relative to numerical data and full line to analytical ones).

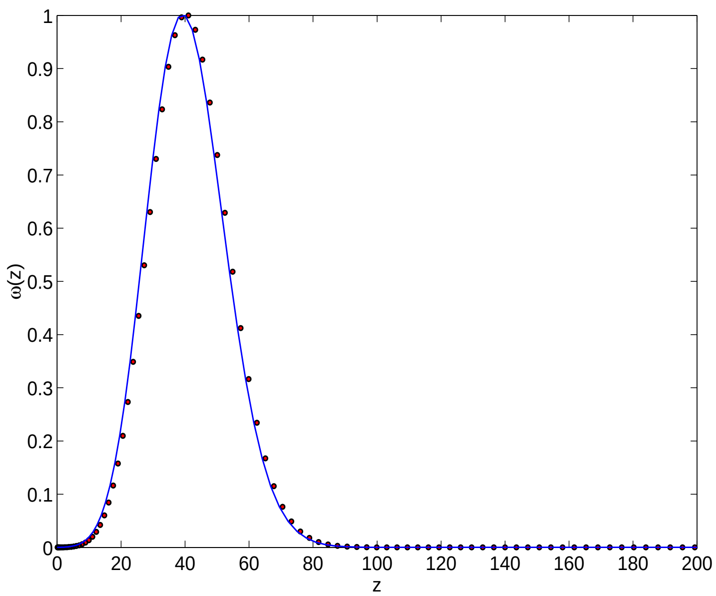

Figure 2: Numerical and analytical first global eigenfunctions

As far as the case of the equation with no-cross-wise effects, the relative computed and analytical spectra are depicted in fig. 3, that shows again a good agreement of data as well as decreased properties of stability of the flow. Finally, the spectrum relative to the more

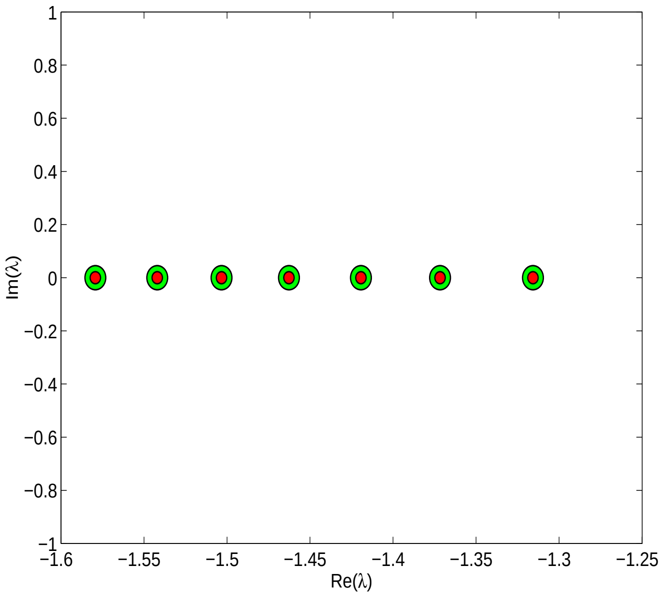

Figure 3: Spectrum in the absence of the crosswise effect

physical situation for which the first derivative coefficient in eq. (9) is taken equal to $\sqrt{1+\delta z}$ is presented in fig. 4, for both cases of $n = 1$ and $n = 5$, and $Re = 1$, $\delta = 10^{-2}$, $\beta = 1$. These spectra have to be compared with that shown in fig. 1. No significant qualitative differences seem to occur. Furthermore, considering the first crosswise mode $n = 1$ seems to be more safe in ascertaining the global stability properties.

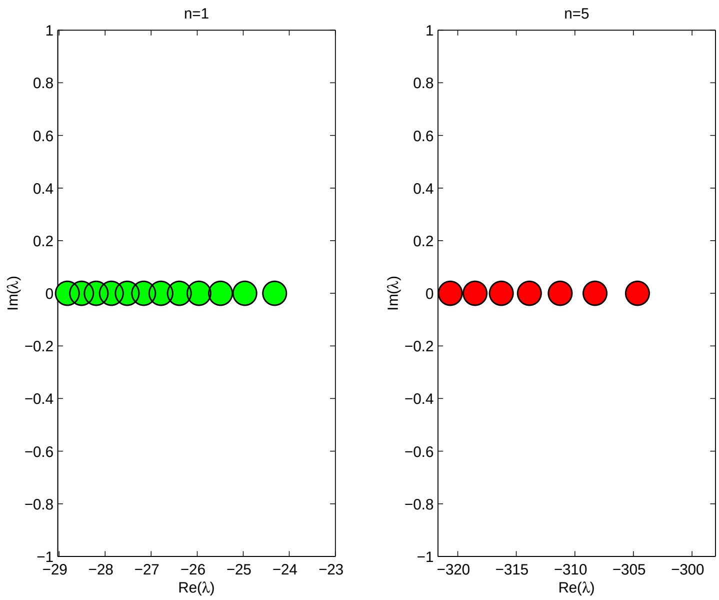

Figure 4: Spectrum of the complete equation ($n = 1$ and $n = 5$)

5 Conclusions

This paper is aimed to formulate the problem of the global stability of a plane liquid sheet flow falling vertically under gravity effects. Due to the necessity of assessing the validity of the overall model, the analysis has been limited to the investigation of the vorticity equation only. In spite of such limitations, the present work approaches the effects on the stability properties of the crosswise variations of the disturbances, neglected in previous studies of the literature.

Preliminary results show that the crosswise effects increase the properties of stability of the global modes. The overall procedure has been tested by means of a rather severe comparison between numerical and analytical data obtained with exact solutions in simplified cases.

6 References

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