The two vortex sheets model in the description of jet instabilities

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The two vortex sheets model for the study of jets instability is discussed. Sinuous waves are shown to resemble flapping waves on an elastic membrane, except for a pressure gradient which acts as a restoring or a destabilising force, depending on the relative magnitude of the wave and the external fluid velocity. The absolute or convective nature of possible instabilities is highlighted by inspecting the large time behaviour of the system Green's function, which also allows to show lacks coming from the use of normal modes. Surface tension free models are demonstrated to be not consistent with the limit of the Weber number going to infinity, since they fail in introducing a length characteristic for wave propagation and in satisfying the causality property for the system response to an introduced perturbation.

1. Introduction

Disintegration of liquid jets and the consequent atomisation process are of major importance to the operation of devices used on a daily basis in a wide variety of fields. Control of injection of liquid fuels is the key to smooth, efficient and low emissions operation of Diesel and gasoline direct injection engines. In the aerospace industry, the engine thrust, efficiency and the emissions level are directly related to the performance of the liquid fuel injector design. Electrostatic car-body spray painting, agricultural drop spraying, ink jet printing, pharmaceutical nebulizers, spray drying and chemical liquid rockets are further examples of technologies involving liquid sprays.
At present, design and control of liquid injectors rely for the most on an empirical basis and strongly suffer of the lack of a solid theoretical background allowing to predict the jet behaviour and instability in a gaseous atmosphere.
Present paper is devoted to perform a basic study of the instabilities arising at the interface of a liquid jet, and to show artefacts coming from the use of simplified jet models and from the trivial application of traditional methods of analysis, as those based on normal modes.
After considering the problem in connection with the wave propagation on a membrane, the dispersion relation of sinuous and varicose waves on plane liquid jets is analysed, in order to search for temporally growing perturbations (temporal modes). As an alternative, the method based on the evaluation of the system Green's function is applied.

1.1 The two vortex sheet model

When uniform streams of two different fluids separated by a thin plate merge together downstream of the plate trailing edge, viscous diffusion promotes strong velocity gradients and high concentration of vorticity in what is called the mixing layer. The basic velocity

profile exhibits there at least one inflection point, the presence of which, according to the Rayleigh's criterion, is typically responsible of instabilities of inviscid nature.
The Rayleigh's criterion only applies to profiles continuous in the cross stream direction. Piecewise velocity profiles, on the other hand, are of more immediate approach, since vorticity can be considered as concentrated on a vortex sheet separating the two fluids. These are dealt with within a potential theory, and the vortex sheet remains as a zero-thickness interface where to enforce the kinematic condition on the particle displacement and the dynamic condition expressing the pressure jump as a function of surface tension.
When two vortex sheets at a distance aside are considered, one resorts to the simplest model to study the instability of two-dimensional jets. This prototype of flow and related interfacial phenomena are of great importance. Typical frequencies and wavenumbers of unstable perturbations, in fact, both scale on the jet width and not on the thickness of the individual mixing layer originating at the nozzle exit, thus implying that the reason for the instability has to be searched in the interaction of the two mixing layers. The phenomenon is, therefore, suitable to be modelled by bringing to zero the mixing layers thickness, and considering two-vortex sheets at a distance apart equal to the jet width. Sinuous perturbations to the undisturbed configuration are recovered if the vortex sheets move in the same direction, varicose perturbations correspond to a displacement in the opposite direction.
Although plain, this description hides some interesting aspects, which prevent using it trivially.
Founder studies about liquid jets flowing into a gas, usually called sheets or curtains, are those of Squire [1] and Taylor [2] who first derived the dispersion relation governing wave propagation. A non zero density of the external fluid was immediately shown to be a necessary condition for instability by means of a temporal stability analysis performed within the classical normal modes approach.
Solutions of equations governing stability of any given system in terms of temporal modes, so as of spatial modes, allow expressing the response of the system itself to a perturbation as their superposition. It is evident that this kind of analysis does not consider that the response, being an effect, can not anticipate the producing cause, namely, in a temporal sense, if a perturbation is assumed to be introduced into the system at the time $t=0$, the response to this perturbation has to be equal to zero for $t<0$. Normal modes, indeed, are said to not satisfy *causality*, and the solution of an initial value problem remains the only way to correctly study the phenomenon.
Since the response of a given system to any perturbation can be expressed as convolution product between the so-called system Green's function and the source term representing the perturbation, interest needs to be moved from the analysis of temporal or spatial modes to the determination of the Green's function, which is the system response to a perturbation in the form of an impulse. Moreover, in defining stability or instability, one has to take into account that, even if the determined response has the form of an unstable perturbation, it may be or advected by the flow itself or propagate both downstream and upstream at once. In the first case the *convective* instability outstrips any spatial location and substantially leaves the medium in its unperturbed state (in the absence of further forcing), whereas, in the second situation, the *absolute* instability makes for growth in time to persist everywhere.
Yu and Monkewitz [3] first explored the effect of the fluids density on the convective or absolute nature of the instability of two-dimensional jets and wakes. Their surface tension free model showed a density lower for the fastest fluid to promote an absolute instability, while a density lower for the slowest one to have the opposite effect. Since a link was shown to exist between the absolute instability promotion and the appearance of self-excited oscillations in jets, so as between the absolute instability suppression and the von Karman

vortex street disappearance in wakes, detection of regions of absolute and convective instability in the space of the governing parameters has a fundamental importance in order to achieve control of these flows.

Studies about liquid sheets in a gas atmosphere, aimed to discover the absolute or convective nature of instabilities, were performed by Lin *et al.* [4], de Luca and Costa [5] and Teng *et al.* [6]. The possible action of gravity was shown to lead to a typical spatially evolving flow, where regions of absolute and convective instability follow one another.

2. Theoretical formulation of the stability problem

A two-dimensional jets is considered as an internal flow surrounded on both sides by an unbounded external one. A Cartesian reference frame *xy*, as sketched in fig. 1a, is defined. The origin is set on the symmetry axis of a nozzle (slit) whose thickness is $2b^*$, and whose extension in the direction orthogonal to the *xy* plane is much more greater than b^*. Hereafter star denotes dimensional quantities, and subscripts *1* and *2* variables relative to the internal and external flow, respectively. Two vortex sheets are assumed to be placed in correspondence of interfaces. Both flows are irrotational. Density and velocity have a *top hat* profile. Velocity is drawn in fig. 1a. Possible deformations, in the form of sinuous or varicose perturbations, are sketched in fig. 1b.

Be ρ^*_1 and ρ^*_2 the constant densities of the two fluids and u^*_1 and u^*_2 the uniform and only non zero components of the velocity vectors. The Weber number and the density ratio are defined as

$$We = \frac{\rho_1^* u_1^{*2} b^*}{\sigma^*} \qquad\qquad r = \frac{\rho_2^*}{\rho_1^*}. \tag{1}$$

Wavelike perturbations are considered, namely,

$$\Psi' = \hat{\Psi} e^{ikx - i\omega t}. \tag{2}$$

For simplicity, the internal dimensionless velocity is assumed to be equal to the unity in a reference frame moving with the external fluid, whereas the dimensionless velocity of this last is taken equal to *a*. This allows describing jet with co-flow, as shown in fig. 2a, with $a \geq 0$, and jet with counter-flow, as shown in fig. 2b, with $-0.5 \leq a \leq 0$.

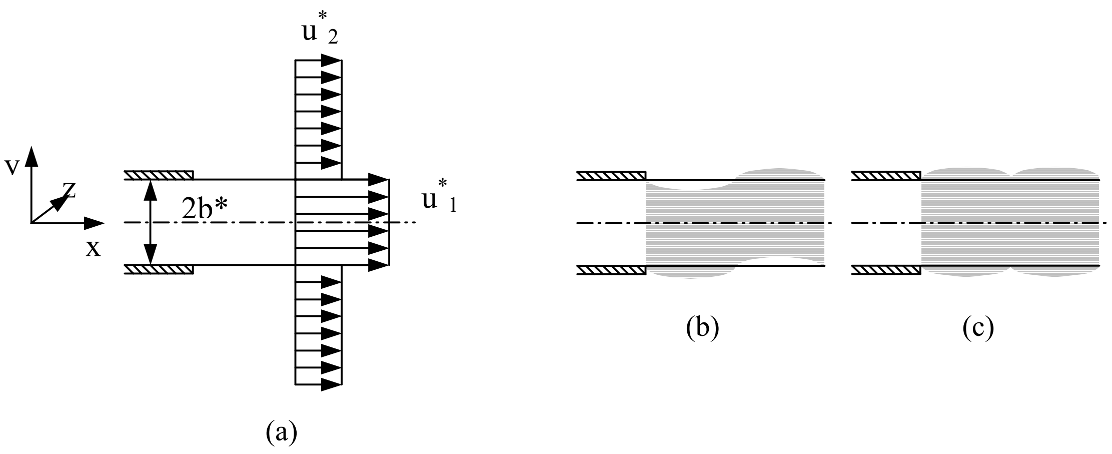
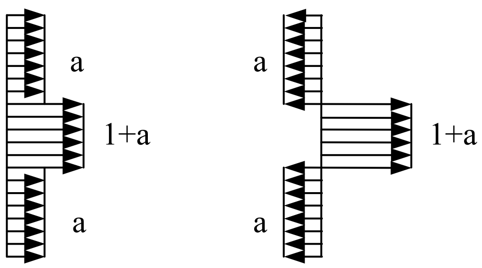


Fig. 1 Sketch of (a) the unperturbed jet flow and of the of the perturbed configuration with (b) sinuous waves and (c) varicose waves.



(a) (b)

Fig. 2 Velocity profile for two different values of the external flow parameter *a*.
(a) $a \geq 0$: jet with co-flow. (b) $-0.5 \leq a \leq 0$: jet with counter-flow.

If a=0 the well known dispersion relation between the wavenumber k and the frequency ω is:

$$\frac{(\omega - k)^2}{\omega^2} = -f(k)\left(r - \frac{k^3}{\omega^2 We}\right), \tag{3}$$

where

$$f(k) = \frac{e^k + se^{-k}}{e^k - se^{-k}}, \tag{4}$$

and $s=1$ ($s=-1$) for sinuous (varicose) waves. Eq. (3) formally coincides with the one derived by de Luca and Costa [5] for a liquid sheet falling under gravity in a still atmosphere. The variables have there a local character and the result is obtained to the lowest order approximation in a small parameter related to the characteristic length of the flow evolution in the streamwise direction.
Generalisation to keep into account the motion of the external fluid is directly achieved by introducing the following transformation:

$$\omega \to \omega - ka \tag{5}$$

which also gives, as Doppler effect, the link between the wave frequency in the reference frame singled out by the slit and the wave frequency in a reference frame moving with the external fluid. The following equation is derived:

$$\left[\frac{\omega - (1 + a)k}{\omega - ka}\right]^2 = -f(k)\left[r - \frac{k^3}{(\omega - ka)^2 We}\right] \tag{6}$$

where $f(k)$ is still given by eq. (4).

2.1. Temporal stability analysis

If $r=0$ is taken, only solutions to both eqs.(3) and (6) in terms of ordinary waves are found for real k. The parameter a only enters to modify the real part of ω, and does not affect stability. The neutral curve, i.e. the locus of $\omega_i=0$ in the k, We plane, which summarises results of the temporal stability analysis, is, indeed, the same for all the values of a. Figs. 3a and 3b show neutral stability curves of sinuous and varicose waves, respectively, for different values of r. Complex frequencies with positive imaginary parts, indicating the existence of temporal instabilities, are found as solution of the dispersion relation for k and We belonging to the region internal to the curve. The cut-off wavenumber for which instability is present gets smaller as r decreases, becoming equal to zero for $r=0$, in agreement with the previously made observation that neglecting the external fluid density implies only ordinary waves to be found. For both varicose and sinuous modes the vertical axis in the diagram does not belong to the instability region. For $k=0$, $\omega=0$, at each value of the Weber number.

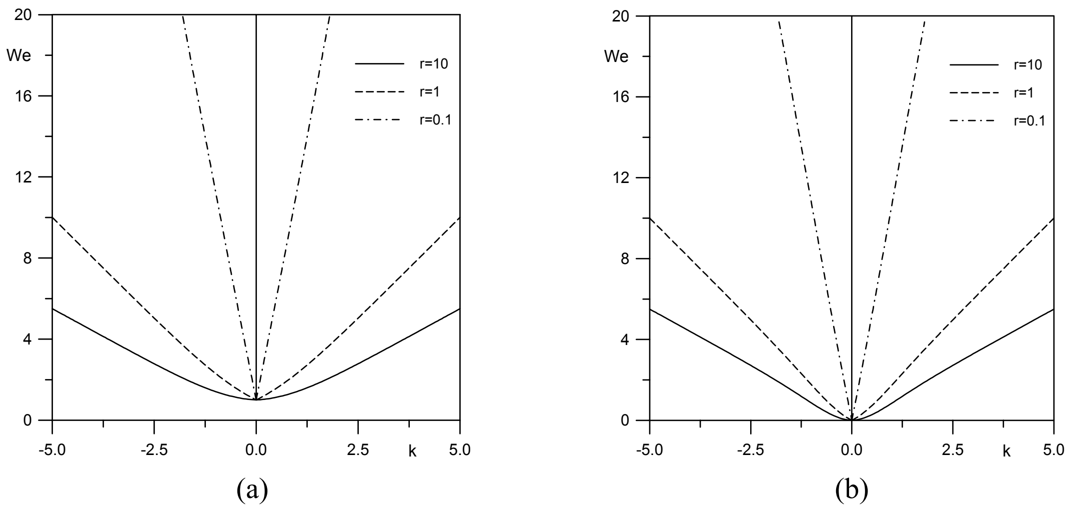


Fig. 3 Neutral stability curve of (a) sinuous waves and (b) varicose waves.

Independently of the value of a, when sinuous modes are considered, there exists a limit value of We below which no complex root of ω exists. This critical value is neither affected by the gas-to-liquid density ratio. The existence of exponentially growing sinuous waves only in the range $We>1$ can be related to the magnitude of the velocity sinuous waves exhibit with respect to the current. Contrary to varicose waves, sinuous waves have a minimum speed which is a function of We. Since the current has been taken with a dimensionless velocity equal to the unity, each sinuous wave is stabilised when moving faster than the current.
In order to clarify the concept, the sheet can be regarded as "frozen", namely as it was a flag, or an inelastic membrane of constant thickness, subjected to a wind. Due to the change in the reference frame the wind has a negative unitary velocity with respect to the membrane. A small value of r is considered, so to perform a perturbation analysis with respect to the case of $r=0$. Within this last situation, ordinary waves, which only satisfy the dispersion relation, have a constant phase velocity, equal to $We^{-1/2}$, in the limit as k tends to zero. In the same limit, which is consistent with observation of fig. 3, eq. (3) takes the form:

$$\omega^2 = \frac{k^2}{We} + rk\left(\frac{1}{We} - 1\right). \qquad (7)$$

Going back to the differential counterpart of eq.(7), so to have:

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{1}{We}\frac{\partial^2 \eta}{\partial x^2} + ir\left(\frac{1}{We} - 1\right)\frac{\partial \eta}{\partial x} \qquad (8)$$

it is clear that the term

$$\Delta p = ir\left(\frac{1}{We} - 1\right)\frac{\partial \eta}{\partial x} \qquad (9)$$

proportional to $(1/We-1)$ clearly adds to what is the equation of flapping waves on a membrane, as hereafter derived.

2.2. Wave propagation on a membrane

An initially flat membrane of density ρ_m is considered, on the ends of which constant tension forces are applied. It is assumed to lie, say, on the *xz* plane of a right handed co-ordinate system *x, y, z*, on which two dimensional waves, with straight wave crests lines orthogonal to the *xy* plane, are considered. End effects are neglected.

A perturbation to the initial configuration, that is a local displacement of a small amount, leads to wave motion, whereas tension acts to restore the initial configuration. Even if initially an immediate return is delayed by the inertia of the displaced portion of the membrane, after a while the momentum acquired by this portion causes it to overshoot the initial position. As time progresses the local perturbation spreads along the whole membrane.

To formally describe the phenomenon, a point on the membrane is taken, which in the unperturbed state is assumed at *(x, 0)* and is displaced, by perturbation, to some point *A* specified by the vector

$$\underline{r} = \xi \underline{i} + \eta \underline{j} . \tag{10}$$

A neighbouring point at *(x+dx, 0)* gets the position *B* whose displacement vector is

$$\underline{r} + \underline{dr} = (\xi + d\xi)\underline{i} + (\eta + d\eta)\underline{j} . \tag{11}$$

Newton's second law of dynamics is considered, considering that in the perturbed configuration the net vector force acting on a unitary length of the element *AB* is the vector sum of the force $\underline{\tau}$ and the force $\underline{\tau} + (\partial \underline{\tau}/\partial x)dx$ at B, and that the mass of the element *AB* of the membrane is the same it has initially, namely $\rho_m dx$:

$$\frac{\partial \underline{\tau}}{\partial x} = \rho_m \frac{\partial^2 \underline{r}}{\partial t} . \tag{12}$$

We assume that the direction of $\underline{\tau}$ is given by the unit vector tangent to the membrane $\underline{d\hat{s}}$, where $\underline{ds}$ is:

$$\underline{ds} = (dx + d\xi)\underline{i} + d\eta \underline{j} \tag{13}$$

whose magnitude

$$ds = \left[(dx + d\xi)^2 + d\eta^2\right]^{1/2} = \left[\left(1 + \frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{1/2} dx \tag{14}$$

represents the length of the element *AB*. The magnitude of $\underline{\tau}$ is related to the tension acting on the undisplaced membrane, σ_o: indeed, at any position σ_o is incremented by an amount proportional to *(ds-dx)/dx*, owing to elasticity. The relation:

$$\tau = \sigma_o + \sigma' \frac{ds - dx}{dx} \tag{15}$$

is assumed to hold. It follows that

$$\underline{\tau} = \left[\sigma_o + \sigma'\left(\frac{ds}{dx} - 1\right)\right]\underline{d\hat{s}} . \tag{16}$$

Considering small values of $\partial \xi/\partial x$ and $\partial \eta/\partial x$, linearisation can be performed, so that $\underline{\tau}$ is re-written as:

$$\underline{\tau} = \left(\sigma_o + \sigma' \frac{d\xi}{dx} \right) \underline{i} + \sigma_o \frac{\partial \eta}{\partial x} \underline{j} . \tag{17}$$

Consequently eq.(12) separates into the scalar equations:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho_m}{\sigma'} \frac{\partial^2 \xi}{\partial t^2} \qquad\qquad \frac{\partial^2 \eta}{\partial x^2} = \frac{\rho_m}{\sigma_o} \frac{\partial^2 \eta}{\partial t^2} . \tag{18}$$

Two decoupled family of waves originates in the linearised regime. The first equation governs propagation of small amplitude longitudinal waves which are *elastic* waves; the second equation describe small amplitude transverse waves which can be called *flapping* waves. Both elastic and flapping waves have a nondispersive character, being the wave velocities

$$c_e = \frac{\omega}{k} = \sqrt{\frac{\sigma'}{\rho_m}} \qquad\qquad c_f = \frac{\omega}{k} = \sqrt{\frac{\sigma_o}{\rho_m}} \tag{19}$$

only dependent on the nature of the medium constituting the membrane. The analogy of the phase speed of flapping waves and phase speed of sinuous waves on a two dimensional jet is evident, if the second term in the right hand side of eq. (7) is neglected. On the other hand, the meaning of this second term is evident. Its sign, namely the sign of Δp in eq. (9), depends on the relative magnitude of the current speed and what is the phase speed of sinuous waves for $r=0$. This last can be taken to not change due to the small value of r. Thus, the conclusion is drawn that the difference in speed makes for a pressure gradient to arise, which may act either in the same direction as surface tension, preserving the unperturbed configuration, either in opposite direction, so to make waves to grow progressively. In evaluating the sign of the pressure gradient due to the presence of the current, it is not sufficient to only refer to the gradient in the velocity of the fluid passing over the deformed membrane, as a stationary Bernoulli theorem would state, but it is necessary to also consider the role of the non-stationary term due to the interface deformation itself [7].
An analogous reasoning leads to state that the absence of a minimum value of the phase speed of varicose waves, which always remain dispersive, explains the existence of temporally growing waves whatever is the value of the Weber number, always existing a wave moving slower than the current being destabilised.

3. Absolute/convective instability analysis

Present paragraph is devoted to show major results coming from the determination of the system Green's function and from the study of the absolute or convective character of the instabilities. It is shown that this kind of approach highlights what is the influence on stability of the parameter a. Indeed, curves separating regions of absolute instability from regions of convective instabilities are drawn in the a, We parameter space for given values of r.
The criterion is applied according to which the flow is absolutely (convectively) unstable if $\omega_{0i}>0$ ($\omega_{0i}<0$), being ω_{0i} the imaginary part of the complex frequency ω_0 for which a pinch point k_0 occurs in the complex k plane by coalescence of two distinct spatial branches $k^+(\omega)$ and $k^-(\omega)$ lying, for large enough $\omega_i>0$, into the half-planes $k_i>0$ and $k_i<0$, respectively [8]. Results are represented for two different values of r in fig. 4, for both sinuous and varicose modes. The grey zone represents the occurrence of absolute instabilities for r=1. Outside the grey zone convective instabilities are found, except for sinuous modes with $We \leq 1$.

The transition curves between absolute and convective instability are obtained by following pinch points corresponding to absolute instability regions as a is increased, and determining the value of this parameter for which the pinch traverses the real axis of the complex k plane. Note that values of $a \leq -0.5$ are of no interest for jet, but correspond to velocity profiles typical of wakes behind bluff bodies, as also reported in ref. [9].

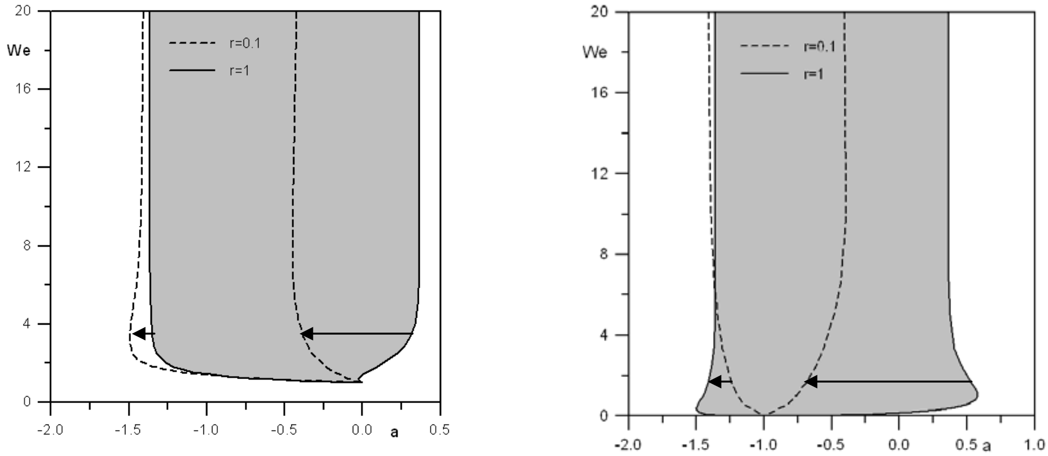


Fig. 4 Regions of convective and absolute instability in the a-We parameters space for sinuous modes (left) and varicose modes (right).

After a $k^+(\omega)$ branch is found by exploiting the fact that any point on the unstable temporal branch $\omega(k)$ also belongs to $k^+(\omega)$, being in particular the intersection of this branch with the real k axis, ω_i is decreased from positive values towards zero and the deformation of $k^+(\omega)$ is followed. When a pinch point is achieved, the branch $k^-(\omega)$ is drawn so to check if it really originates in the negative half-plane. The following scenario appears as the computation is performed. Once a pinch with $\omega_{0i}>0$ is found, further deformation of $k^-(\omega)$ shows the existence of a second pinch, which indeed persists even when a crosses the value corresponding to transition from absolute to convective instability. Moreover, if surface tension is decreased until completely removed, the first pinch, say P_1, moves towards greater and greater values of the wavenumber, which ultimately goes to infinity, whereas the second pinch, P_2, persists at finite values of k, thus indicating it to be due to a typical Kelvin-Helmholtz instability. Both P_1 and P_2 are in fact found in the whole range of parameters considered, and followed as a is varied. For each value of the Weber number, P_1 is higher than P_2 (in the sense that it is formed for an higher value of ω_i) in an interval of values of a, whose extent depends on the Weber number itself. Within this interval the system exhibits an absolute instability. Outside P_2 is higher than P_1, but in any case it can not be taken to dominate the system response since it always originates from branches both originating in the same half-plane.

As Weber is increased, not appreciable variations are observed in the limit values of a for which transition from absolute to convective instability occurs. For $r=1$, as an example these values are $a=-1.365$ and $a=0.365$. Present results are not in agreement with those of Yu and Monkewitz [3]. Indeed, ref. [3] considers a surface tension free model. In the situation where surface tension is brought to zero, as when two miscible fluids are considered, perturbations on each interface do not feel the presence of the other interface, and the system behaves as the distance between the two vortex sheets was infinite. Indeed, the absence of surface tension makes for a cut-off wavelength for instability to disappear. This is also evident as one looks at fig. 5. As the Weber number is increased, ω_i ultimately becomes a monotone function of k_r, and satisfying causality, for an infinite Weber number, becomes not possible.

Indeed, no way exists to perform integration in the complex plane, always existing a value of ω_i higher than that corresponding to any value of k. This leads to conclude that surface tension free models are not suitable to be employed for describing jet instability as a consequence of an introduced perturbation, within the initial value problem approach.

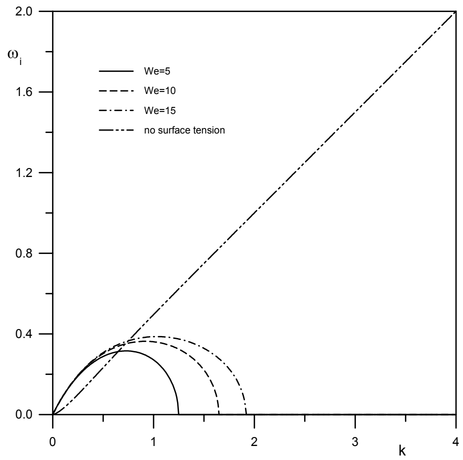


Fig. 5 Neutral stability curve of varicose waves: (a) $r=10$, (b) $r=1$, (c) $r=0.1$.

A point must be stressed as regards the system behaviour with respect to sinuous perturbations with a Weber number less than the unity. The case $a=0$ was indeed considered in ref. [5]. Even if no complex ω are found as solution for real k, the region of absolute convergence for the Green's function can not be extended below the real axis of the complex ω plane. A pinch point occurs just at the origin, where, on the other hand, the dispersion relation behaves as:

$$(\omega - \omega_o)^2 \approx (k - k_o)^3 \tag{20}$$

being $\omega_0=k_0=(0,0)$. It follows that $G(x,t)$ is a power function of time with an exponent equal to one third. This corresponds to an algebraic growth for G, the order of the singularity dominating the Green's function, more than its position, making for the existence of an absolute instability region [10, 11].

Finally, it has to be considered that when the jet thickness suffers variations along the streamwise direction (non-parallel flows), there may exist a point along this direction where the Weber number crosses the unity or the parameter a crosses the value corresponding to the transition from convective to absolute instability. The case of a sheet evolving in the radial direction is one in which the Weber number decreases from values greater to values smaller than the unity, as getting farther from the origin of the sheet. The transition point is a limit for the size of the sheet, since rupture of the flow can be related to the occurrence of the absolute instability [12]. If a falling sheet is considered, on the contrary, acceleration of the fluid in the vertical direction, and decrease of the thickness, make the Weber number eventually to grow from values less than the unity to values greater. A region of absolute instability may be present close to the slit exit section, thus explaining the rupture of the sheet observed in reducing the flow rate [5]. Indeed the existence of a region of absolute instability is a necessary but not sufficient condition for the flow to break up. Being the greater the extension of this region the smaller the flow rate, rupture of the sheet may be believed to be due to the presence of a region of absolute instability of sufficiently great extent.

4. Conclusions

An asymptotic study of the two vortex sheets model allows to explain the different behaviours of jets to sinuous or varicose perturbations. Stability or instability of sinuous waves to exponentially growing modes is related to the gradient of pressure acting normal to the vortex sheets, which strongly depends on the non-stationary term one has to consider in applying the Bernoulli's theorem correctly. The existence of a minimum phase speed for sinuous waves, which is not the case for the varicose ones, explains why waves moving faster than the current are stabilized and the existence of a limit value of the Weber number below which no exponentially growing perturbations are possible.
The determination of the large time behaviour of the system Green's function, on the other hand, besides stating the absolute or convective nature of possible instabilities, puts light on the singular behaviour the two vortex sheets model exhibits in the limit as the surface tension is brought to zero. For a given value of the external to the internal fluid density the interval of values of the external fluid velocity for which an absolute instability is found is shown to be underestimated if a surface tension free model is employed. Neglecting this quantity makes for a cut-off wavelength for instability to disappear and artefacts to be needed in order to satisfy causality. The validity of these is not confirmed by the results we obtained as the surface tension goes to zero in a model fully including its effects.

5. References

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