Spray characteristics from truncated Lévy statistics

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We show using a simple cascade modelling that log-stable volume distribution are a theoretical alternative to usual empirical drop spray distributions. We then recall their better fitting to experimental drop spray distributions in a horizontal annular gas-liquid flow experiment. They also better model moments and especially negative moments of the distribution. It is then shown that they are, consequently, more adequate to compute the Sauter mean diameter or other characteristics of the spray related to negative moments of the volume distribution, such as the spray intensity or the spray surface density.

1. Introduction

In this work, we show how log-stable p.d.f. can be used to model spray drop distributions and characteristics as Sauter mean diameter, spray intensity or surface density for instance. There is a wide range of p.d.f used to describe spray p.d.f.; from Rosin-Rammler to Nukiyama-Tanasawa or upper-limit lognormal, none has a real physical basis [1]. An exception is Kolmogorov's derivation in 1941 of log-normal number p.d.f to describe size of particles under pulverisation. Thought it is a less known paper than his 1941 turbulence theory [2], it was however the root of the theory of intermittency in turbulence [3]. Later, Novikov [4] applied his work on infinitely divisible distributions in turbulence back to spray drop size. Recently Kolmogorov's work has been revisited and extended to continuous time by Gorokhovski and Saveliev [5].

We will firstly show how a simple cascade mechanism can lead to a log-stable volume p.d.f. Lévy stable laws are a generalisation of Gaussian normal law and a peculiar case of infinitely-divisible distribution. Actually, these laws are much easier to handle in practical computations than infinitely divisible laws since their characteristic functions has a simpler closed form. We will recall several of our recent results [6] of the fitting of log-stable law to experimental p.d.f. obtained by Simmons and Hanratty in a horizontal annular gas-liquid flow [7]. We point out the better performance of the log-stable distribution over empirical logarithmic distributions by showing their ability to fit to the moments of the experimental distribution. However, this modelling involve negative moments of the distribution, which are rigorously infinite. Accordingly, the distributions need to be truncated but they are, then, much more difficult to handle analytically. We show next that, since the Sauter mean diameter (SMD) is related to a negative moment of the distribution, log-stable volume p.d.f. happen to be an efficient way to compute the SMD. This will also be the case for other characteristics of the spray related to negative moments of the volume distribution, such as its intensity or its surface density.

2. Simple cascade mechanism and log-stable p.d.f.

2.1. Simple modelling of the atomisation of a high Weber spray.

We will assume that every droplets divide itself into two daughter droplets on a Cayley tree

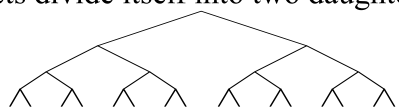

Figure 1: Multiplicative Cascade on a Cayley Tree

The volume of the final droplet can be calculated simply by:

$$V_f = \frac{V_f}{V_{f-1}} \frac{V_{f-1}}{V_{f-2}} \cdots \frac{V_2}{V_1} V_i \tag{1}$$

where V_i is the initial volume and V_k, $k = 1, \ldots, f$, the volume after the k^{th} division. This multiplicative process can be transformed into an additive one using the logarithm:

$$Y = \ln\left(\frac{V_f}{V_i}\right) = \sum_{k=i}^{k=f} \ln\left(\frac{V_k}{V_{k-1}}\right) = \sum_{k=i}^{k=f} \ln(M_k) = \sum_{k=i}^{k=f} X_k \tag{2}$$

Taking into account a division process implies that

$$0 < M_k < 1 \text{ and } -\infty < X_k < 0. \tag{3}$$

At each step in the tree, we can assume that the fragmentation of the drops is mainly governed by its kinetic energy and that both viscous and surface tension effects are negligible. Actually, we can assume that the Weber number, defined by

$$We = \frac{\rho U^2 L}{\gamma} \to \infty$$

and that the Ohnesorge number, defined by

$$Oh = Z = \frac{\sqrt{We}}{\mathrm{Re}} = \frac{\mu}{\sqrt{\sigma\rho D}},$$

is small or bounded. This universality assumption is equivalent to assuming that every drop is de facto unstable and will divide according to the same probability. Otherwise, this probability should be Weber and Ohnesorge dependent [8] Therefore, the random variables X_k are identically and independently distributed.

The atomisation process has now been simply modelled by a sum of random variables. One could use the classical central limit theorem and obtain that the logarithm of the volume of the drop would be normally distributed. However the convergence of the sum of random variables to the so-called normal distribution supposes that variances of the X_k shall be finite. There is no a priori reason to do so and we apply instead a generalised central limit theorem [9, 10] leading to Lévy stable distribution. These stable distributions (of whom Gaussian distributions are simply a special case) are defined as [9]:

A random variable X is said to have a stable distribution if there are real parameters $0 < \alpha \leq 2$, $0 < \gamma$, $-1 \leq \beta \leq 1$ and δ such that its characteristic function has the following form:

$$\hat{p}_\alpha(k, \beta, \gamma, \delta) = \exp\left(k\delta - \gamma^\alpha |k|^\alpha \left(1 + i\beta \operatorname{sign}(k)\omega(|k|, \alpha)\right)\right) \tag{4}$$

where $$\omega(|k|, \alpha) = \begin{cases} \tan(\alpha\pi/2) & \text{if } \alpha \neq 1 \\ -(2/\pi)\log|\mathrm{k}| & \text{if } \alpha = 1 \end{cases}$$

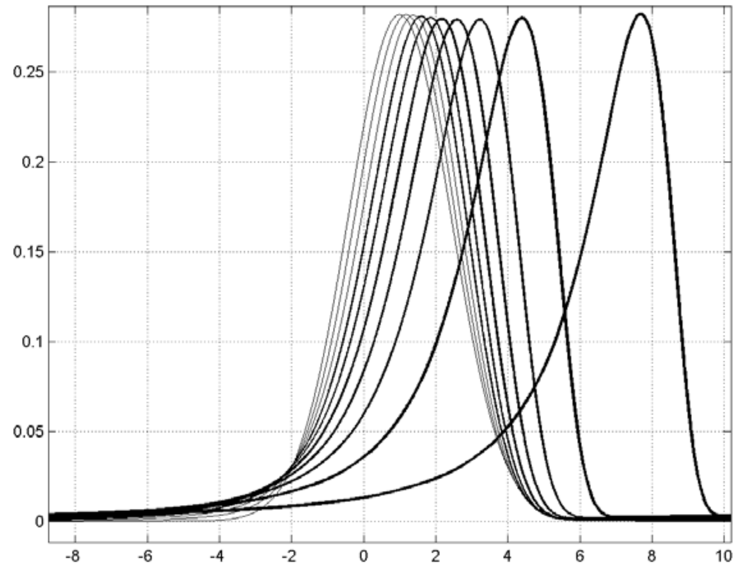

Figure 2: example of stable distributions. From right to left, α is ranging from 1.1 to 2, $\beta = -1$, $\gamma = 1$ and $\delta = 1$.

A representation of some stable p.d.f. is given in figure 2 and the convergence theorem is given by:

[The cumulative] distribution F(x) belong to the domain of attraction of a stable law with the characteristic exponent $\alpha, \left(0 < \alpha \le 2\right)$*, [if and only if:]*

(i) [let] $\mu(x) = \int_{-x}^{x} y^2 F\{dy\}$*, it is necessary that :* $\mu(x) \underset{\infty}{\sim} x^{2-\alpha} L(x)$ *where L [is a slowly varying function at* ∞*.]*

(ii) if $\alpha = 2$*, then F belongs to the domain of attraction of the normal distribution.*

if $\alpha < 2$ *and* $\frac{1-F(x)}{1-F(x)+F(-x)} \to p$ as $x \to \infty$, $\frac{F(-x)}{1-F(x)+F(-x)} \to q$ as $x \to \infty$ *then F belongs to the domain of attraction of [the] stable law [with parameters* α *and* β*. Skewness parameter* β *is related to p and q through:* $\beta = \frac{p-q}{p+q}$*]* (adapted from [10] p544 and [9] p6)

If F is the cumulative distribution function of X_k, (3) implies that $F(x) = 1$ for $x > 0$ and:

$$\frac{1-F(x)}{1-F(x)+F(-x)} = \frac{0}{F(-x)} \text{ which implies that } p = 0$$

$$\frac{F(-x)}{1-F(x)+F(-x)} = \frac{F(-x)}{F(-x)} \text{ which implies that } q = 1$$

This leads to the important fact that β should be equal to -1. It is also possible to construct an inverse cascading mechanism: let $M_k > 1$ thus $0 < X_k < \infty$ and the same reasoning leads to $\beta = 1$. Actually letting $M_k \in [0, \infty[$ may lead to any value of β. This could be a crude modelling of collisions.

2.2. Experimental results

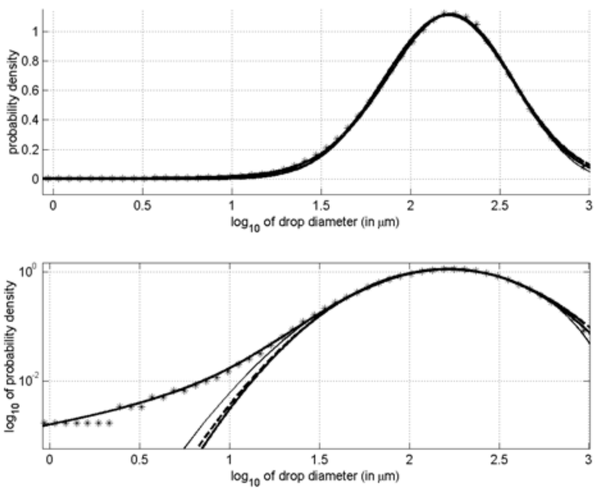

Figure 3: fitting of the drop spray p.d.f. $V_{SL} = 0.041$ m/s $V_{SG} = 35$ m/s. From left to right the p.d.f. are log-stable p.d.f., Upper-limit Evans p.d.f. , log-normal p.d.f. and log-Weibull p.d.f. [6]

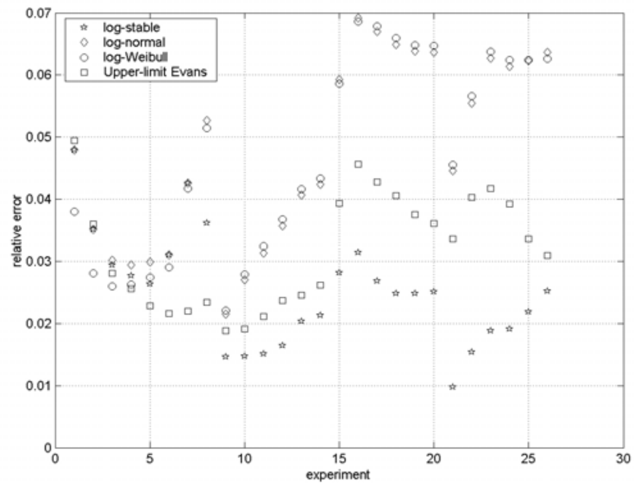

Figure 4:Comparison of the relative error for the different fitted p.d.f. Note that the velocity of the gas is increasing with the index number of the experiment. [6]

We considered p.d.f. measured by Simmons and Hanratty [7] in a horizontal annular gas-liquid flow. In this experimental set-up, the flow loop is an acrylic cylinder 27 m long and $D_w = 0.0953$ m wide. Mass flows and velocities of air and water are metered at the entrance and measurements are made 21 m away from the entrance, so that the flow can be considered as fully developed. Drops diameters D are measured using a Malvern Spraytec RTS 5008 analyser. Data were processed on billions of drop giving very smooth p.d.f. over three decades of drop diameters (ranging from less than $1\ \mu m$ to $1000\ \mu m$). Important parameters of the experiments are V_{SL} the velocity of the liquid relatively to the solid cylinder, ranging from 2.2 to 13.5 cm/s, V_{SG} the velocity of the gas relatively to the solid, ranging from 30 to 50 m/s, and the entrained liquid mass flux G_{LE} i.e. the amount of water in the drops cloud (related to V_{SL} and V_{SG} in an intricate though tabulated way). From these external parameters, one can calculate three important dimensionless numbers [1]

the global Weber Number

$$We = \frac{\rho_L V_{SG}^2 D_w}{\gamma} \approx 3.10^6 ,$$

the global Ohnesorge number

$$Oh = \frac{\mu}{\sqrt{\gamma \rho_L D_w}} \approx 3.10^{-6} ,$$

the dimensionless liquid concentration

$$\phi = \frac{G_{LE}}{\rho_L V_{SG}} \approx 10^{-4} ,$$

where γ is the surface tension of the liquid, ρ_L its density and μ its dynamic viscosity.

The p.d.f. were reported using the drop volume distribution, which is given by:

$$\frac{dV}{dD} = f_V(D)\,, \tag{6}$$

$$\int_0^\infty f_V(D).dD = 1\,, \tag{7}$$

$$f_V(D).dD = f_n(D).D^3.dD \tag{8}$$

where f_n is a non normalized number distribution. Actually, $\int_0^\infty f_n(D).dD$ is the number of drops in a unit volume of fluid. It can be seen that:

$$\Pr\left\{V_f = \frac{\pi}{6}D^3\right\} = \frac{\pi}{6}\frac{D^3 f_n(D)}{\int_0^\infty f_n(D).dD} = \frac{\pi}{6}\frac{f_V(D)}{\int_0^\infty D^{-3} f_V(D).dD}$$

and since the number of drops in a unit volume can be assumed to be constant at any measurement point; the volume distribution can be assumed to follow a log-stable law.
This stable p.d.f. is calculated by the inverse Fourier transform of (4) and is then fitted by minimizing the error function:

$$Err_2 = \sqrt{\frac{\sum\left(\tilde{y}_i - y_i\right)^2}{\sum y_i^2}} \tag{9}$$

where y_i are the measurements and $\tilde{y}_i$ the results of the fitting. Figure 1 shows the result of such a fitting and figure 2 the evolution of Err_2 for 26 experimental results situated at the centre-line of the pipe. All four parameters α, β, σ, δ were free in order to give the best fit to the experimental data. Comparison is made between log-stable, Upper-limit Evans, log-Weibull and log-normal volume p.d.f. On the lower diagram the p.d.f. has been represented on a logarithmic scale putting an emphasis on the tail of the distribution. It is clear that log-stable laws tackle better with the left tail of these p.d.f. This could be foreseen since for $\beta = -1$ et $1 < \alpha < 2$ one gets the following asymptotic results

$$p_\alpha(x) = \begin{cases} \approx \exp\left(-Ax^{\frac{\alpha}{\alpha-1}}\right) & \text{if } x \to \infty \\ \approx |x|^{-(\alpha+1)} & \text{if } x \to -\infty \end{cases},$$

with

$$A = \frac{\alpha - 1}{\alpha^{\alpha/(\alpha-1)}\left(-\sigma\Big/\cos\left(\frac{\pi\alpha}{2}\right)\right)^{1/(\alpha-1)}}$$

We can notice that in the large drop part, the p.d.f. behaves like a stretched exponential and is very close to the Weibull p.d.f. α was found to be located in the range]1.75, 2] and, with a typical value of $\alpha = 1.8$, one gets a value of 2.25 for the stretching exponent. The value $\beta = -1$ was found in twenty-four of the twenty-six experiments and was very close to -1 in the remaining two cases. We can consider that it is not a free parameter any longer and the log-stable p.d.f. offer as many degrees of freedom as empirically constructed p.d.f. i.e. three parameters.

3. Moments and spray characteristics

3.1. Moments

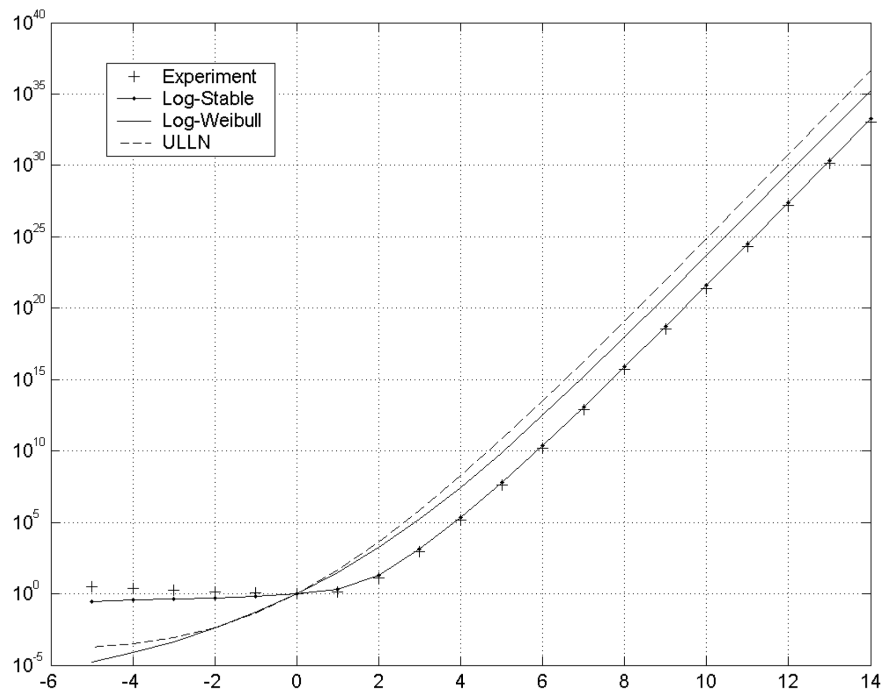

Figure 5 moment <D^q> of the drop spray p.d.f. Comparison between experimental and several fitted p.d.f. $V_{SL} = 0.022$ m/s $V_{SG} = 43$ m/s. [6]

Moments are defined from the normalized number distribution as

$$\left\langle D^q \right\rangle = \frac{\int D^q f_n(D).dD}{\int f_n(D).dD} = \frac{\int D^{q-3} f_V(D).dD}{\int D^{-3} f_V(D).dD}$$

In figure 5, a comparison between moments of the experimental data and of several fitted distributions is made. The very poor results of empirical p.d.f. is linked to their inability to model negative moments of the volume distribution. Their inadequacy is enhanced by the fact that all the results are normalized by the moment of order minus three. These moments were computed with integration made from the smaller drop size measured to the larger drop size. This introduce a cut-off in the distribution. This is important since log-stable p.d.f. are diverging in zero like

$$f_V(D) \approx \frac{1}{D} \left|\ln(D)\right|^{-(\alpha+1)} \text{ if D} \to 0$$

and negative moments without cut-off are infinite and thus are not defined. For positive moments without cut-off, one gets the following analytical formulae [9]:

$$\int D^q f_V(D).dD = \int \exp\left(q \ln(D)\right).p_\alpha\left(\ln(D)\right).d\ln(D) = \mathcal{L}\left(p_\alpha\right)(q) = \exp\left[q\delta - \frac{\sigma^\alpha}{\cos\left(\frac{\pi\alpha}{2}\right)} q^\alpha\right]$$

where $\mathcal{L}$ is the two-sided Laplace transform and q is a positive number.

3.2. *Sauter mean diameter*

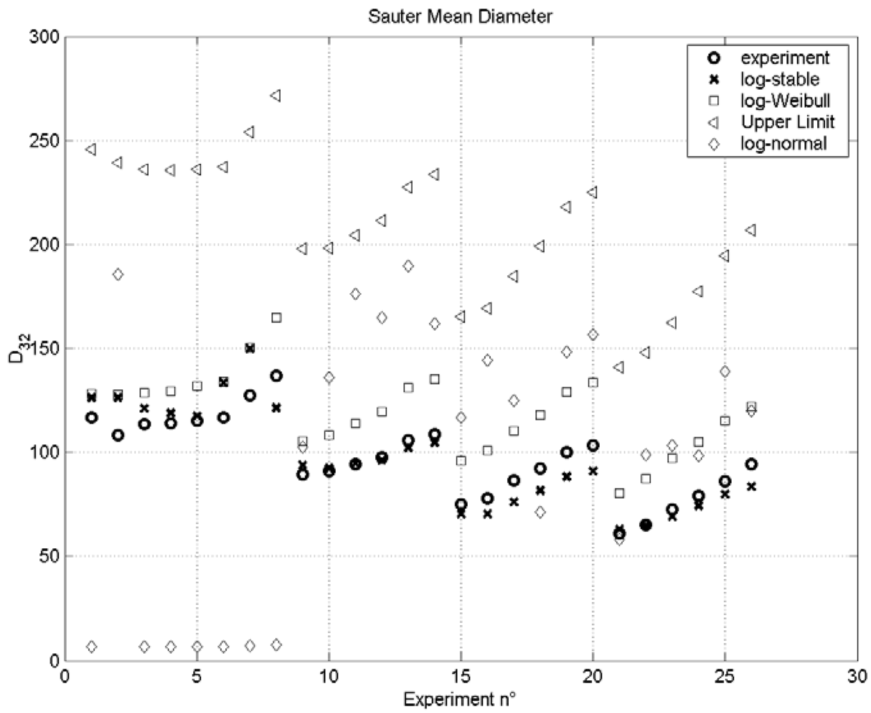

Figure 6: Sauter mean diameter. Comparison between experimental results and values given by several fitted p.d.f.

Let us recall firstly that the definition of the Sauter mean diameter involves a negative moment of the volume distribution :

$$d_{32} = \frac{\int D^3.f_n(D).dD}{\int D^2.f_n(D).dD} = \frac{\int f_V(D).dD}{\int D^{-1}.f_V(D).dD} = \frac{1}{\int D^{-1}.f_V(D).dD}$$

Since truncated log-stable distribution fit well to moments and especially negative moments of the volume distribution, this leads to strong assumptions that they could provide a good way to calculate the Sauter mean diameter or other characteristics of the spray. This is verified in figure 6.

3.3. *Spray intensity*

Other characteristics of the spray stem from negative moments of the distribution. The spray intensity can be defined as the mean number of droplets by unit volume of the mixture and is thus given by

$$n = \phi \int D^{-3} f_V(D).dD\,.$$

This definition also involves a negative moment of the volume distribution and is thus very sensitive to the small drop part of the distribution.

3.4. Surface density

The surface density is the mean area of the interfaces between the gas phase and liquid phase by unit volume of the mixture and is thus given as

$$\Sigma = \pi.\phi \int D^{-1} f_V(D).dD = n.\pi.d_{20}^2,$$

since

$$d_{20} = \left(\frac{\int D^2 . f_n(D).dD}{\int f_n(D).dD} \right)^{1/2} = \left(\frac{\int D^{-1} . f_V(D).dD}{\int D^{-3} . f_V(D).dD} \right)^{1/2} .$$

4. Conclusion

In this work, we have shown how a simple cascading mechanism can be used to predict a log-stable volume p.d.f for the drop size in a spray. We have recalled some recent results on the fitting of such distributions to experiments conducted by Simmons and Hanratty [7]. We have then analysed the sequels of this results i.e. the possibility for the truncated log-stable distributions to be used to calculate moments of the distribution and to accurately "predicts" the value of the Sauter mean diameter. We have then shown that they could also be used in conjunction with the dimensionless liquid concentration, a.k.a. volume of fluid, to compute other characteristics of the spray: its intensity or its surface density.

The present modelling is however static and forthcoming work will include time dependence. This has already been done for the log-normal distribution through a non-linear Fokker-Planck equation [5]; and for the log-stable distribution, through a dedicated master equation [6]. The main problem of this approach is related to the need to truncate Lévy distribution so as to be able to compute their negative moments. Forthcoming works will cope with this problem and propose transport equations for the different spray characteristics based on this approach.

5. Acknowledgement

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6. Reference

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