The Analogy between Swirl Atomizer and Weir Flow: The Principle of Maximum Flow

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In a number of early papers presenting a simple mathematical, analytical treatment of swirl atomizers the diameter of the air-core within the outlet was determined by the use of the principle of maximum flow. This was by analogy to the principle of maximum flow as applied to weirs. However, in none of these works the “Principle of Maximum Flow” either cited or proven for the swirl atomizer. This paper demonstrates that the principle of maximum flow can indeed be applied, in theory, to swirl atomizers under the assumptions made by the previous workers.

1. Introduction

Before the advent of Computational Fluid Dynamics (CFD) many workers attempted to determine the air-core diameter within a swirl atomizer by simple mathematical, analytical means. Once the air-core diameter in the outlet region was determined it was then a simple matter to calculate the axial velocity, the swirl velocity and the discharge coefficient. It was then possible to make an estimate of the spray cone angle. The accuracy of these parameters, derived within these works, was limited in comparison to CFD and experiment, due to their lack of sophistication. For example, they did not include viscous or turbulent effects. However such simple mathematical models do allow us to better understand the fundamental dynamics and forces at work within the swirl atomizer [1, 2, 3, 4].

For instance, Giffen and Muraszew formulated an equation involving both the discharge coefficient (a measure of the volumetric flow rate for a given liquid density, operating pressure and outlet orifice diameter) and the air-core diameter in the outlet as unknowns. Having one equation and two unknowns is an unsolvable mathematical system. In order to alleviate this problem they invoke the ‘principle of maximum flow’. In essence this means that the air-core diameter will adjust itself so that the volumetric flow rate will be a maximum. Following the usual method of determining a maximum or a minimum, the equation is rearranged so that the discharge coefficient is a function of the air-core. This is then differentiated and set equal to zero. One then obtains one equation in one unknown, the air-core diameter in the outlet orifice.

However, in all of these works the principle of maximum flow for a swirl atomizer is assumed. It has never, to the best of the present authors belief, been demonstrated. This paper sets out to show the formulation of the principle of maximum flow for a swirl atomizer by analogy to the standard textbook derivation for the principle of maximum flow for a weir.

Weirs are essentially small, submerged walls placed across a river in order to step the river down a hillside and therefore control its flow velocity and hence prevent undue erosion. The water rises from a deeper level, on the upstream side of the weir and then flows over the top of the weir, to then cascade down over the downstream side. The volume flow over the crest of the weir is of course the same as it is in the deeper upstream region of the river. However, the flow is faster due to the reduced cross-sectional area of the water. The remarkable phenomenon is that the surface level of the water is actually lower, over the top of the weir, than it is in the main body of water in the upstream region. This height, and the flow velocity, adjust so that the flow rate is a maximum.

In a swirl atomizer the swirl chamber is analogous to the deeper region upstream of the weir and the outlet corresponds to the weir itself. It is also found that the liquid surface level changes between the upstream region and the outlet. In other words the air-core diameter, and the axial velocity, adjust so that the flow rate is a maximum.

In weir flow the surface topology and dynamics are governed by the volumetric flow rate and the force of gravity. In swirl atomizer flow the surface form and dynamics are governed by the volumetric flow rate and the analogous centrifugal force.

2. Swirl Atomizer Analysis

The flow domain for this analysis is a convergent/divergent nozzle shown in figure 1, where the flow is moving from left to right. The differential between the stagnation pressure in some supply reservoir and that at the air-core, at which the pressure is zero gauge, is Δp. The usual supposition regarding the radial velocity v is that it is negligible in comparison to the swirl and axial velocities (w and u, respectively). The radial velocity v, of the bulk flow, will certainly be small adjacent to the air-core.

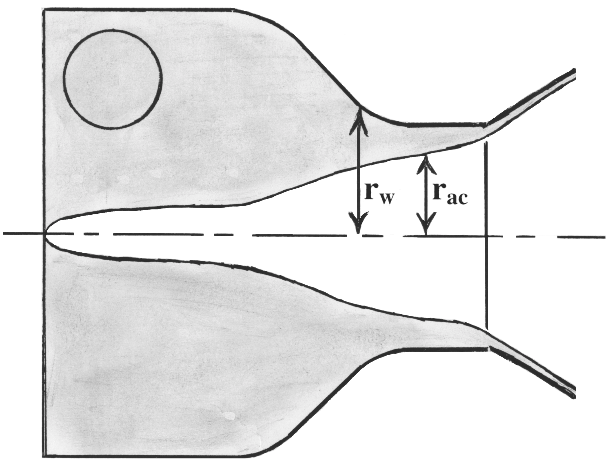

Figure 1. Cross-Section of a typical swirl atomiser showing the air-core and the variables: r_w, the wall radius, and r_{ac}, the air-core radius. For the purpose of this analysis, the corner between the convergence and outlet is shown as a radius. In practice there is likely to be an effective radius there due to boundary layer effects.

The analysis is in two parts, as is the treatment of weir flow, in the appendix, which should be compared with the analysis given here. In the first part, an expression for the axial velocity u_c through the 'throat' or constriction (in practice, the outlet) is determined by putting the axial change in the volumetric flow equal to zero, $\partial Q/\partial x = 0$, by continuity. In the second part an expression is derived for $\partial Q/\partial r_{ac}$ at the throat; where r_{ac} is the radius of the air-core at any axial position, x, within the nozzle. It is then shown that by putting the expression for u_c (the 'critical velocity'), obtained in the first part, into this expression for $\partial Q/\partial r_{ac}$, obtained in the second part, that indeed $\partial Q/\partial r_{ac} = 0$ at the throat. Thus, the air-core in the throat will adjust itself so as to permit maximum flow. This is then a direct analogy of the flow over a weir where the water depth h, over the crest of the weir, adjusts itself so that the flow is a maximum.

2.1 First Part

The first part begins with the Bernoulli equation for this flow, which is

$$\frac{u^2}{2} + \frac{w^2}{2} + \frac{p}{\rho_L} = \frac{\Delta p}{\rho_L}. \tag{1}$$

Here ρ_L is the liquid density. By continuity, the mean axial velocity across any cross-section of the nozzle will be given by

$$u = \frac{Q}{\pi\left(r_w^2 - r_{ac}^2\right)}. \tag{2}$$

For irrotational flow, the swirl velocity is given by a free vortex so that, at the air-core within the nozzle:

$$wr = w_{ac}r_{ac} = c \tag{3}$$

At the air-core, p = 0 so that on using eqn.(3) for w_{ac} then eqn.(1) may be rearranged to form an expression for u:

$$u = \left(\frac{2\Delta p}{\rho_L} - \frac{c^2}{r_{ac}^2}\right)^{1/2}. \tag{4}$$

Next, eqn.(2) is arranged to make Q the subject and is then differentiated w.r.t. x, the axial coordinate, to form

$$\frac{\partial Q}{\partial x} = 2\pi u\left(r_w\frac{dr_w}{dx} - r_{ac}\frac{dr_{ac}}{dx}\right) + \pi\left(r_w^2 - r_{ac}^2\right)\frac{\partial u}{\partial x} = 0. \tag{5}$$

This is zero, as the volumetric flow rate Q is continuous throughout the nozzle. Equation (4) may also be differentiated w.r.t. to x to form

$$\frac{\partial u}{\partial x} = \left(\frac{2\Delta p}{\rho_L} - \frac{c^2}{r_{ac}^2}\right)^{-1/2}\frac{c^2}{r_{ac}^3}\frac{\partial r_{ac}}{\partial x} \quad \text{or} \quad u\frac{\partial u}{\partial x} = \frac{c^2}{r_{ac}^3}\frac{\partial r_{ac}}{\partial x}. \tag{6}$$

Equation (5) may be rearranged to make dr_{ac}/dx the subject and the resulting expression may be substituted for dr_{ac}/dx into eqn.(6) to give

$$u\frac{\partial u}{\partial x} = \frac{c^2}{r_{ac}^3}\left(\frac{\left(r_w^2 - r_{ac}^2\right)}{2ur_{ac}}\frac{\partial u}{\partial x} + \frac{r_w}{r_{ac}}\frac{dr_w}{dx}\right). \tag{7}$$

The grouping of the coefficients of du/dx in the above yields

$$\frac{\partial u}{\partial x}\left(\frac{u^2 - u_c^2}{u}\right) - \frac{c^2 r_w}{r_{ac}^4}\frac{dr_w}{dx} = 0, \tag{8}$$

where u_c is defined as

$$u_c^2 = \frac{c^2\left(r_w^2 - r_{ac}^2\right)}{2r_{ac}^4}. \tag{9}$$

At the throat (outlet) $\partial r_w/\partial x = 0$, but $\partial u/\partial x \neq 0$ because the annulus of liquid continues to become thinner with x as it passes through the outlet (as shown in figure 1) which in turn causes u to continue to increase. Therefore, in order for eqn.(8) to be equal to zero then it only remains that the axial velocity u must be equal to u_c, as given in eqn.(9), where r_w and r_{ac} take their respective values at the throat: $r_w(x_c)$ and $r_{ac}(x_c)$.

2.2 Second Part

The second part of this analysis is designed to demonstrate the principle of maximum flow for a nozzle by putting $\partial Q/\partial r_{ac} = 0$; this parallels the weir flow analysis where $\partial Q/\partial h$ is set equal to zero. Equation (1) is again rearranged to make Q the subject and the resulting expression is differentiated, this time w.r.t. r_{ac}, giving

$$\frac{\partial Q}{\partial r_{ac}} = -2u\pi r_{ac} + \pi\left(r_w^2 - r_{ac}^2\right)\frac{\partial u}{\partial r_{ac}}. \qquad (10)$$

Equation (4) is also differentiated w.r.t. r_{ac} which results in

$$\frac{\partial u}{\partial r_{ac}} = \left(\frac{2\Delta p}{\rho_L} - \frac{c^2}{r_{ac}^2}\right)^{-1/2}\frac{c^2}{r_{ac}^3} \quad \text{or} \quad \frac{\partial u}{\partial r_{ac}} = \frac{1}{u}\frac{c^2}{r_{ac}^3} \qquad (11)$$

By substituting this expression for $\partial u/\partial r_{ac}$ into eqn.(10) one obtains, on simplification,

$$\frac{\partial Q}{\partial r_{ac}} = \frac{2\pi r_{ac}}{u}\left(\frac{c^2\left(r_w^2 - r_{ac}^2\right)}{2r_{ac}^4} - u^2\right). \qquad (12)$$

At the throat u must be equal to u_c, as given by eqn.(9), which was derived by the sound reasoning that the volumetric flow, by continuity, will not vary with axial distance, $\partial Q/\partial x = 0$, and then eqn.(12) gives that, at the throat, $\partial Q/\partial r_{ac} = 0$. Thus the air-core radius at the throat, $r_{ac}(x_c)$, must adjust itself so that Q is a maximum. In practice, as Q is a constant, this means that the air-core radius and the axial velocity within the outlet will adjust themselves optimally. This is the principle of maximum flow, which has been used without this manner of qualification by the authors cited.

3. Discussion

It can be shown [5] that eqn.(18) below, $u_c = \sqrt{(gh)}$, is in fact both the critical velocity over the crest of the weir and a simplified expression for the wave phase velocity of a long shallow water gravity wave. Similarly, eqn.(9) is both an expression for the critical velocity within the outlet of the swirl atomizer and the wave phase velocity of a shallow ‘centrifugal wave’ occurring on the surface of the air-core. Nieuwkamp[4] evidently had some realisation of this fact in his mathematical swirl atomizer analysis.

4. References

[1] Taylor G I 1948. “The Mechanics of Swirl Atomizers”. International Congress of Applied Mechanics, London, Volume 2, Part 1;

[2] Giffen E and Muraszew A 1953. “Atomization of Liquid Fuels”. Chapman and Hall.

[3] Bayvel L and Orzechowski Z 1993. “Liquid Atomization”, Taylor and Francis.

[4] Nieuwkamp W C (1985) “Flow Analysis of a Hollow Cone Nozzle with Potential Flow Theory”. Proc. ICLASS-85 London IIIC/1-9.

[5] Chinn J J 2003. “The Analogy between Waves on the Surface of the Air-core of a Swirl Atomizer and Long, Shallow Water, Gravity Waves”. ICLASS-2003 Sorrento Italy.

5. Appendix: Weir Flow Analysis

In the flow over a weir an expression for the critical velocity at the crest of the weir may be formulated by the assumption that the volume flow will not vary with the direction of streaming. The expression for the critical velocity is then used to show that the height of the water over the crest of the weir will always be such as to permit the most efficient flow rate.

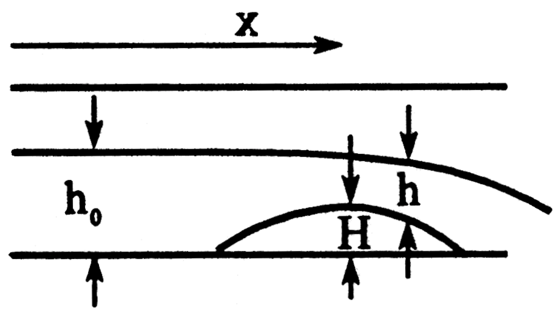

Figure 2. Flow, from left to right, of Liquid over a Weir.

The flow takes place in a channel, of width B. The liquid is maintained at a constant height h_0 above the level bottom. The weir is depicted, in figure 2, as a hump placed on the bottom of height H(x). This submerged wall is here shown curved, in practice this is likely to be an accurate representation due to boundary layer effects. The depth of the liquid, from the free surface down to the weir is h(x). The constant velocity for the liquid in the main channel is denoted u_0 and that flowing over the weir u(x). The methodology works by first determining an expression for the streaming velocity of the liquid passing over the **c**rest of the weir u_c (the 'critical velocity') and then, secondly, by obtaining an expression for $\partial Q/\partial h$ into which is substituted this expression for u_c. It is then found that this gives $\partial Q/\partial h = 0$ at the crest of the weir showing, through the usual method of determining a maximum or minimum, that the depth of liquid h(x), at the crest, will be such as to permit maximum flow.

5.1 First Part

The first part of this analysis, to determine the velocity over the crest of the weir, begins with the Bernoulli equation which, for the free surface streamline of this flow, is

$$h(x) + H(x) + \frac{u^2(x)}{2g} = h_0 + \frac{u_0^2}{2g}. \tag{13}$$

The volumetric flow rate Q, over the weir, is given by:

$$Q = uhB \tag{14}$$

where B is the breadth of the channel. Here it is assumed that the channel has parallel sides so that B does not vary either in the vertical direction or the direction of streaming, x. The differentiation of eqns.(13) and (14) w.r.t. x give

$$\frac{\partial h}{\partial x} + \frac{dH}{dx} + \frac{u}{g}\frac{\partial u}{\partial x} = 0 \tag{15}$$

and

$$\frac{\partial Q}{\partial x} = Bu\frac{dh}{dx} + Bh\frac{\partial u}{\partial x} = 0 \tag{16}$$

which is equal to zero as Q will not vary with x, the direction of streaming. Both eqn.(15) and (16) may be rearranged to make $\partial h/\partial x$ the subject. The resultant expressions are equated with one another to provide

$$\left(\frac{u}{g} - \frac{h}{u}\right)\frac{\partial u}{\partial x} + \frac{dH}{dx} = 0. \tag{17}$$

At the crest of the weir, $dH/dx = 0$. The streaming velocity u(x) will continue to increase with axial direction as it passes over the weir so that $\partial u/\partial x \neq 0$, at the axial position of the crest of the weir. If both of these, reasonable, assumptions are applied to eqn.(17) then it only remains that, at the crest,

$$u = u_c = \sqrt{gh}, \tag{18}$$

where the subscript c represents the crest, or critical position. This establishes an expression for the velocity of the flow over the crest of the weir and this will be used presently.

5.2 Second Part

The second part of this weir flow analysis is designed to establish the principle of maximum flow for a weir i.e. that the volumetric flow rate Q is a maximum as a function of the height h of the weir by showing that $\partial Q/\partial h = 0$ at the weir crest. By rearranging eqn.(13) to make u the subject and employing eqn.(14) one may derive

$$Q = \sqrt{2gh^2 B^2 \left(h_0 + \frac{u_0^2}{2g} - h - H \right)}. \tag{19}$$

So that, as B, h_0 and u_0 are constants and H is not a function of h, then

$$\frac{\partial Q}{\partial h} = \frac{1}{2}\left(2gB^2h^2\left[h_0 + \frac{u_0^2}{2g} - h - H \right] \right)^{-1/2} 2gB^2\left\{ 2h\left[h_0 + \frac{u_0^2}{2g} - h - H \right] - h^2 \right\}. \tag{20}$$

From eqn.(13)

$$h_0 + \frac{u_0^2}{2g} - h - H = \frac{u^2}{2g} \tag{21}$$

so that, after some work, eqn.(20) simplifies down to

$$\frac{\partial Q}{\partial h} = \frac{B}{u}\left(u^2 - gh \right). \tag{22}$$

Thus if $u = u_c = \sqrt{(gh)}$ at the crest of the weir, as derived in eqn.(18), then indeed $\partial Q/\partial h = 0$. This establishes that the volumetric flow rate Q is a maximum as a function of h at the crest of the weir or, as by continuity Q is clearly a constant, that the height and velocity of the flow over the hump, h(x) and u(x), adjusts themselves optimally.